A. Zero temperature

Susceptibility can be easily calculated at $T = 0$ for any momentum and frequency. Here for simplicity we drop the frequency.

$$\chi_0(q) = \frac{1}{2\pi}\sum_k \int d\omega \frac{1}{(i\omega - \varepsilon_{k+q})(i\omega - \varepsilon_k)} \tag{5}$$

$$= -2\sum_k \frac{\Theta(-\varepsilon_k)}{\varepsilon_k - \varepsilon_{k+q}} \tag{6}$$

$$= \frac{1}{2\pi^2}\int \frac{\Theta(-\varepsilon_k) k dk d\phi}{kq\cos\phi + q^2/2} \tag{7}$$

$$= \frac{1}{\pi^2}\int \frac{\Theta(-\varepsilon_k) k dk d\phi}{kq(z+z^{-1}) + q^2} \tag{8}$$

$$= \frac{1}{\pi^2}\int_0^{k_F} \frac{dk}{q}\int_C \frac{dz/i}{z^2 + zq/k + 1}. \tag{9}$$

Here C is the unit circle and $z = e^{i\phi}$. The poles of the integrand are $z_\pm = -q/(2k) \pm \sqrt{q^2/(2k)^2 - 1}$. For $k < q/2$ the z_+ pole is inside unit circle and z_- outside. Therefore,

$$\chi_0(q) = \frac{1}{\pi}\int_0^{\min(k_F, q/2)} \frac{dk}{q}\frac{1}{\sqrt{q^2/(2k)^2 - 1}} \tag{10}$$

$$= \frac{1}{\pi q}\sqrt{q^2/4 - k^2}\Big|^0_{\min(k_F, q/2)}, \tag{11}$$

and therefore,

$$\chi_0(q < 2k_F) = \frac{1}{2\pi} \tag{12}$$

$$\chi_0(q > 2k_F) = \frac{1}{2\pi}\left(1 - \sqrt{1 - 4k_F^2/q^2}\right) \tag{13}$$

We see that zero-temperature susceptibility is completely flat for $q < 2k_F$. This remains true (in a reduced interval of momenta) also at finite frequency.

B. Finite temperature

Let us now determine the form of susceptibility at finite temperature, first assuming that $(k_F q, \Omega) \ll T$. For brevity assume that $\Omega = 0$.

Then from Eq. (**??**) after performing summation over ω_n,

$$\chi(q) = -\sum_k \frac{n_F(\varepsilon_k) - n_F(\varepsilon_{k+q})}{\varepsilon_k - \varepsilon_{k+q}} \tag{14}$$

$$= -\sum_k n_F'(\varepsilon_k) + \frac{1}{2}n_F''(\varepsilon_k)(\varepsilon_{k+q} - \varepsilon_k) + \frac{1}{6}n_F'''(\varepsilon_k)(\varepsilon_{k+q} - \varepsilon_k)^2 + ... \tag{15}$$

$$= -\sum_k n_F'(\varepsilon_k) + \frac{1}{2}n_F''(\varepsilon_k)(\mathbf{kq} + q^2/2) + \frac{1}{6}n_F'''(\varepsilon_k)(\mathbf{kq} + q^2/2)^2 + ... \tag{16}$$

$$= -\int \frac{kdkd\phi}{(2\pi)^2}\left[n_F'(\varepsilon_k) + \frac{1}{2}n_F''(\varepsilon_k)(\mathbf{kq} + q^2/2) + \frac{1}{6}n_F'''(\varepsilon_k)(\mathbf{kq} + q^2/2)^2 + ...\right] \tag{17}$$

$$= -\int \frac{d\varepsilon_k}{2\pi}\left[n_F'(\varepsilon_k) + \frac{1}{2}n_F''(\varepsilon_k)q^2/2 + \frac{1}{6}n_F'''(\varepsilon_k)((\varepsilon_k + \mu)q^2 + q^4/4) + ...\right] \tag{18}$$

$$= \frac{1}{2\pi} \tag{19}$$