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FASTER THAN FOURIER

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Written to celebrate the 60th Birthday of Yakir Aharonov: deep, quick, subtle.

ABSTRACT

Band-limited functions $f(x)$ can oscillate for arbitrarily long intervals arbitrarily faster than the highest frequency they contain. A class of integral representations exhibiting these 'superoscillations' is described, and by asymptotic analysis the origin of the phenomenon is shown to be complex saddles in frequency space. Computations confirm the existence of superoscillations. The price paid for superoscillations is that in the infinitely longer range where $f(x)$ oscillates conventionally its value is exponentially larger. For example, to reproduce Beethoveen's ninth symphony as superoscillations with a 1Hz bandwidth requires a signal $\exp\{10^{19}\}$ times stronger than with conventional oscillations.

1. Model for superoscillations

My purpose is to decribe some mathematics inspired by Yakir Aharonov during a visit to Bristol several years ago. He told me that it is possible for functions to oscillate faster than any of their Fourier components. This seemed unbelievable, even paradoxical; I had heard nothing like it before, and learned only recently of just one related paper[1] in the literature on Fourier analysis (see §4). Nevertheless, Aharonov and his colleagues had constructed such 'superoscillations' using quantum-mechanical arguments[2]. Here I will exhibit a large class of them, and use asymptotics and numerics to study their strange properties in detail.

Consider functions $f(x)$ whose spectrum of frequencies k is band-limited, say by $|k|\le 1$, so that on a conventional view f should oscillate no faster than $\cos(x)$. But we wish f to be superoscillatory, that is to vary as $\cos(Kx)$, where K can be arbitrarily large, for an arbitrarily long interval in x. A representation that achieves this is

$$f(x,A,\delta)=\frac{1}{\delta\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathrm{d}u\,\exp\{\mathrm{i}xk(u)\}\exp\left\{-\frac{1}{2\delta^2}(u-\mathrm{i}A)^2\right\} \tag{1}$$

where the wavenumber function $k(u)$ is even, with $k(0)=1$ and $|k|\le 1$ for real u, A is real and positive, and δ is small. Examples are

$$k_1(u)=\frac{1}{1+\frac{1}{2}u^2},\quad k_2(u)=\operatorname{sech}u,\quad k_3(u)=\exp\left\{-\tfrac{1}{2}u^2\right\},\quad k_4(u)=\cos u \tag{2}$$

Aharonov's reasoning (he suggested Eq.(1) with k_4) was that when δ is small the second exponential would act like a 'complex delta-function' and so project out the value of the first exponential at u=iA. Thus f should vary as

$$f \approx \exp\{\mathrm{i}Kx\} \quad \text{where } K = k(\mathrm{i}A) \tag{3}$$

Under the conditions above Eq.(2), k increases from u=0 along the imaginary axis, so that K>1, (and for the given examples can be arbitrarily large), and so corresponds to superoscillations. What follows is a study of the small-δ asymptotics of the integral representing f. As well as justifying Aharonov's argument, this will dissolve the paradox posed by superoscillations, by showing that when x>O($1/\delta^2$) they get replaced by the expected cos(x), and f gets exponentially large.

2. Asymptotics

The aim is to get an asymptotic approximation for small δ to the integral defining f, Eq.(1), which is valid uniformly in x. To achieve this, it is convenient to define

$$\xi \equiv x\delta^2 \tag{4}$$

so that Eq.(1) can be written

$$f\left(\xi/\delta^2, A, \delta\right) = \frac{1}{\delta\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \, \exp\left\{-\frac{1}{\delta^2}\Phi(u,\xi,A)\right\} \quad \text{where } \Phi \equiv \tfrac{1}{2}(u-\mathrm{i}A)^2 - \mathrm{i}\xi\, k(u) \tag{5}$$

For small δ, f can now be approximated by the saddle-point method, that is by deforming the path of integration through saddles u_S of the exponent and replacing Φ by its quadratic approximation near u_S. f is dominated by the saddle with smallest ReΦ. Saddles, whose location depends on ξ (and also A) are defined by

$$\frac{\mathrm{d}\Phi}{\mathrm{d}u} = 0, \quad \text{i.e. } u_s = \mathrm{i}\left[\xi k'(u_s) + A\right] \tag{6}$$

Application of the saddle-point method now gives the main result:

$$f \approx \frac{\exp\left\{\mathrm{i}x\,k(u_s) - \frac{1}{2\delta^2}(u_s - \mathrm{i}A)^2\right\}}{\sqrt{1 - \mathrm{i}x\delta^2 k''(u_s)}} \tag{7}$$

To interpret this formula, it is necessary to understand the behaviour of the dominant saddle as ξ varies.

When ξ<<1, that is x<<δ^{-2}, Eq.(6) gives $u_S \approx iA$, and (7) reduces to Eq.(3); this is the regime of superoscillations. When ξ>>1, that is x>>δ^{-2}, the saddles are the zeros of $k'(u)$; assuming for simplicity that k has a single maximum at u=0 (as in the first three functions in Eq.(2)), this is the only real saddle, and (7) reduces to

$$f \approx \frac{1}{\delta\sqrt{x|k''(0)|}} \exp\left\{ix - \tfrac{1}{4}\pi\right\} \exp\left\{\frac{A^2}{2\delta^2}\right\} \tag{8}$$

This is the behaviour to be expected conventionally, that is on the basis of the frequency content of f; in the infinite range of validity of Eq.(8), f is O(exp$\{A^2/2\delta^2\}$ and so exponentially amplified relative to the superoscillation regime.

As x increases, the saddle moves from iA to 0 along a curved track, illustrated in figure 1. This is the dominant saddle u_S; its track resembles figure 1 for all $k(u)$ of this type that I have studied. There are other solutions of Eq.(6), whose arrangement and motion are complicated and depend on the details of $k(u)$, but they are not dominant and so do not compromise the validity of Eq.(7) as the leading-order approximation to the integral defining f, Eq.(1).

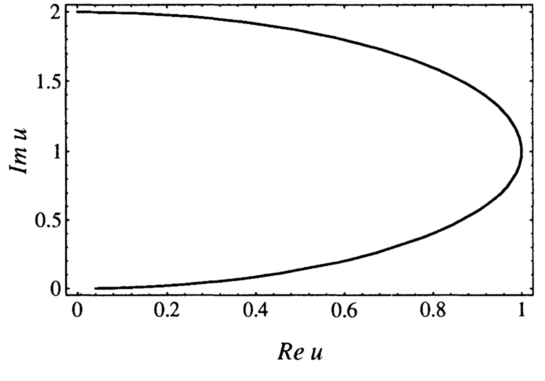


Figure 1. Track of leading saddle u_S as ξ increases from 0 to ∞, for the wavenumber function $k_5(u)$ in Eq.(10), for A=2 (the track is similar for any $k(u)$ with a single maximum)

In understanding the oscillations, it is helpful to study the local wavenumber, defined as

$$q(\xi) \equiv -\mathrm{Im}\frac{\partial\Phi\{u_s(\xi),\xi,A\}}{\partial\xi} = \mathrm{Re}\,k(u_s(\xi)) \tag{9}$$

As illustrated in figure 2, $q(\xi)$ decreases smoothly from k(iA) (which is real) to 1 as ξ increases. Note that the decrease is rapid (this is true for all $k(u)$ that I have studied). This has the important implication that to observe superoscillations it is necessary to keep ξ

much smaller than unity, and if we want to allow x to be large, in order to observe *many* superoscillations, δ must be correspondingly smaller, Eq.(4), and the exponential amplification in the regime of conventional oscillation, Eq.(8), will be correspondingly larger.

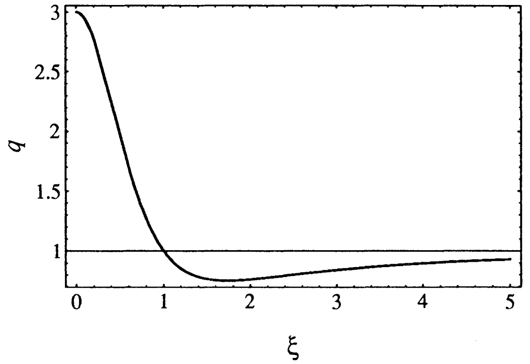


Figure 2. Local wavenumber $q(\xi)$, Eq.(9), for the $k_5(u)$ in Eq.(10), for A=2

None of the wavenumber functions in Eq.(2) gives an f whose integral representation can be evaluated exactly in terms of special functions. However, if we choose the wavenumber function

$$k_5(u) = 1 - \tfrac{1}{2}u^2 \tag{10}$$

we can ensure that it is band-limited ($|k|<1$) by restricting the range of integration in Eq.(1) to $|u|\leq 2$. The resulting truncated integral is

$$f(x,A,\delta) = \frac{1}{\delta\sqrt{2\pi}} \int_{-2}^{2} \mathrm{d}u \exp\left\{\mathrm{i}x\left(1 - \tfrac{1}{2}u^2\right)\right\} \exp\left\{-\frac{1}{2\delta^2}(u - \mathrm{i}A)^2\right\} \tag{11}$$

which be expressed in terms of error functions:

$$\begin{aligned} f(x,A,\delta) = \frac{1}{2\sqrt{1+\mathrm{i}x\delta^2}} \exp\left\{\frac{\mathrm{i}x\left(2 + A^2 + 2\mathrm{i}x\delta^2\right)}{2\left(1 + \mathrm{i}x\delta^2\right)}\right\} \times \\ \times \left[\operatorname{erf}\left\{\frac{2 + \mathrm{i}A + 2\mathrm{i}x\delta^2}{\delta\sqrt{2 + 2\mathrm{i}x\delta^2}}\right\} + \operatorname{erf}\left\{\frac{2 - \mathrm{i}A + 2\mathrm{i}x\delta^2}{\delta\sqrt{2 + 2\mathrm{i}x\delta^2}}\right\}\right] \end{aligned} \tag{12}$$

It is instructive to examine this in detail. The superoscillation wavenumber, Eq.(3), is

$$K = k_5(\mathrm{i}A) = 1 + \tfrac{1}{2}A^2 \tag{13}$$

There is a single saddle, at (figure 1)

$$u_\mathrm{s}(\xi) = \frac{\mathrm{i}A}{1+\mathrm{i}\,\xi} \tag{14}$$

and the local wavenumber is (figure 2)

$$q(\xi) = 1 + \frac{A^2\left(1-\xi^2\right)}{2\left(1+\xi^2\right)^2} \tag{15}$$

For this case, the saddle-point approximation, Eq.(7) gives

$$f(x,A,\delta) \approx \frac{1}{\sqrt{1+\mathrm{i}x\delta^2}}\exp\left\{\mathrm{i}x\left[1+\frac{A^2}{2\left(1+x^2\delta^4\right)}\right]\right\}\exp\left\{\frac{A^2\delta^2x^2}{2\left(1+x^2\delta^4\right)}\right\} \tag{16}$$

However, the asymptotics of (11) includes contributions from the end-points $u=\pm 2$ as well as the saddle u_S. This can be seen by realising that the steepest path between -2 and +2 runs from -2 to infinity in the negative half-plane, through u_S to infinity in the positive half-plane, and back to +2. The end-point contributions oscillate conventionally, with the wavenumber -1, so we must be sure that they do not mask the superoscillations that exist for small ξ. The condition for this is that the absolute value of the Gaussian in (11) must not exceed unity at the end-points. Thus

$$\exp\left\{\frac{A^2-4}{2\delta^2}\right\} \leq 1, \qquad \text{i.e.} \qquad A \leq 2 \tag{17}$$

(we include the equality because the end-point contribution is smaller than that from the saddle by a factor δ). Eq.(13) now implies that the maximum rate of superoscillation obtainable with this model is K=3. (It is worth remarking that x=0, A=2 lies on the anti-Stokes line for the error functions in Eq.(12), that is, where the exponential contribution from the saddle exchanges dominance with those from the end-points.)

The representation Eq.(1) does not have the form of a Fourier transform, namely (for a band-limited function)

$$f(x,A,\delta) = \int_{-1}^{1} \mathrm{d}q\, \exp\{\mathrm{i}xq\}\bar{f}(q) \tag{18}$$

It is however easy to cast it into this form. The transform $\bar{f}(q)$ depends on the inverse function of $k(u)$; this is multivalued, and the path of integration can be deformed into a loop around a cut extending along the real axis negatively from the branch point at q=1 (the ends of the loop are pinned to the cut, at q=-1 for k_5 and at the essential singularity q=0 for k_1, k_2, and k_3). Again there is a dominant saddle, which for small ξ lies at q=K, and the loop can be expanded to pass through this. All previous results can be reproduced in this way.

3. Numerics

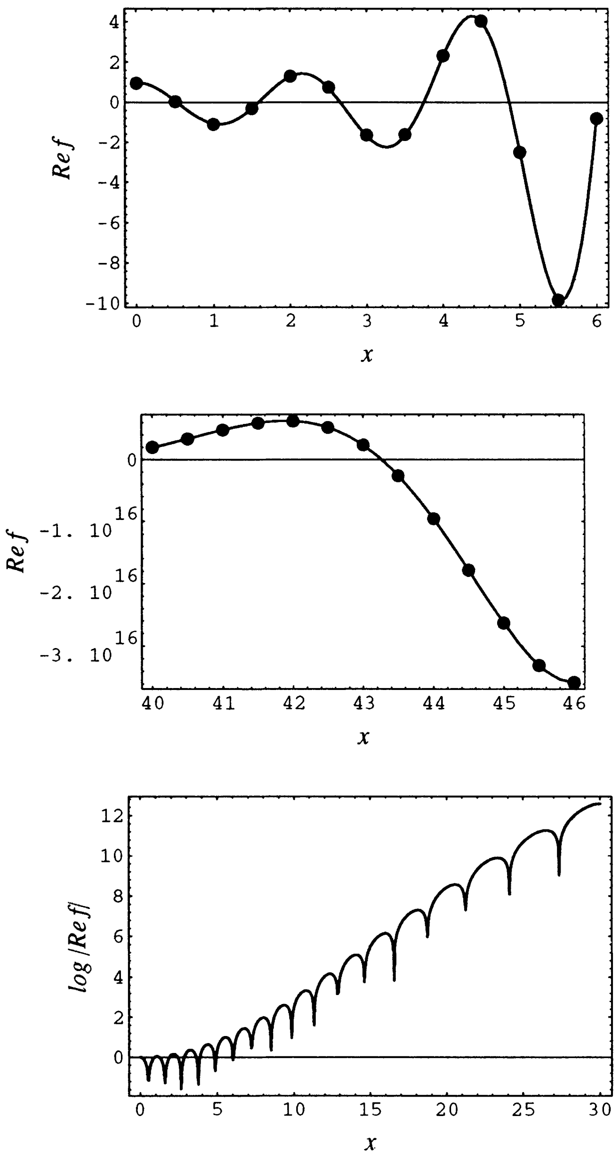
The aim here is twofold: to compare the saddle-point approximation Eq.(7) with the exact integral (1), and to exhibit the superoscillations. I carried out computations of f for the wavenumber functions k_1, k_2, and k_3 (Eq.(2)), but will display results only for Re f (Im f is similar) for k_5 (Eq.(10)), with the truncated integral of Eq.(11), for which the results are very similar. The computations will be exhibited for the fastest superoscillations, namely K=3, that is A=2 (Eq.(17)), choosing δ=0.2.

Figure 3 shows the results. The superoscillations for small x, with period $2\pi/3$, are shown on figure 3a, and figure 3b shows a range of x where there are conventional oscillations, with period more than 3 times greater (actually about 8.4 - cf. figure 2, where $\xi \sim 1.6$ corresponds to $x \sim 40$). In both cases, the approximation (in this case Eq.(16)) agrees well with the exact expression, Eq.(12). For example, the fractional error is 0.18 for x=2, and 2.8×10^{-18} for x=42. Note the enormous ratio of the sizes of f for large and small x; from Eq.(16), this can be estimated as exp(36)~10^{16} (the asymptotic ratio of Eq.(8) is not attained in figure 3b). The transition between the superoscillation and conventional regimes is clearly shown in figure 3c.

In these computations, the value A=2 is the largest for which the saddle dominates the end-points. The competition between contributions shows up most clearly at x=0, for which (12) gives

$$f(0,A,\delta) = \mathrm{Re}\,\mathrm{erf}\left\{\frac{1}{\delta}\left(\sqrt{2} + \mathrm{i}\frac{A}{\sqrt{2}}\right)\right\} \tag{19}$$

For A<2, f is well approximated by the saddle contribution of unity, for A>2, the end-points dominate and f increases exponentially, Eq.(17), masking the superoscillations for small x. This is illustrated in figure 4. Even at the critical value A=2, that is, on the anti-Stokes line for the function (19), the exact value f=0.945 is close to the saddle-point value f~1.



(3a)

(3b)

(3c)

Figure 3. Computations of $f(x,2,0.2)$ for the truncated integral, Eq.(11), showing (a), superoscillations, and (b) conventional oscillations. Circles: exact expression, Eq.(12); full lines: saddle-point approximation, Eq.(16). In (c) the logarithms are base 10

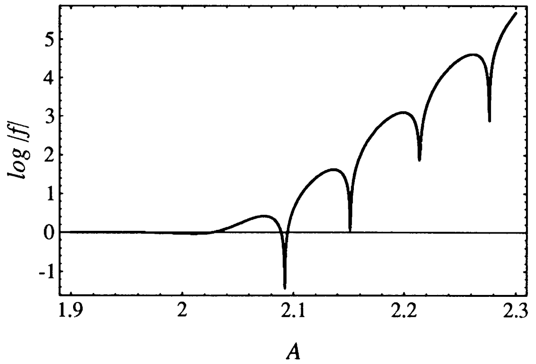


Figure 4. Computations of log $|f(0, A, 0.2)|$, Eq.(19), for the truncated integral Eq.(11); logarithms are base 10. Note the exponential growth after crossing the anti-Stokes line at A=*2*

4. Beethoven at 1Hz

Professor I. Daubechies has informed me that superoscillations are known in signal processing, in the context of oversampling. This is sampling a function faster than the Nyquist rate, i.e. at points $x=n\pi$ where the function is band-limited by $|k|\leq 1$. If a function is oversampled in a finite range, extrapolation outside this range is exponentially unstable[2]. She quotes B. Logan as saying that it is possible in principle to design a bandlimited signal with a bandwidth of 1Hz that would reproduce Beethoven's ninth symphony exactly. With the superoscillatory functions described in this paper it is possible to give an explicit recipe for constructing this signal, as I now explain.

We require superoscillations for the duration T (~4000s) of the symphony. Therefore the desired signal $B(t)$ can be represented as periodic outside this interval, namely

$$B(t) = \sum_{-N}^{N} B_n \exp\left\{ i \frac{2\pi n t}{T} \right\} \tag{20}$$

Here N is the order of the Fourier component corresponding to the highest frequency $\nu_{\max} \equiv N/T$ (~20kHz) it is desired to reproduce.

To approximate this with a signal band-limited by frequency ν_0 (=1Hz) we make the replacement

$$\exp\left\{ i \frac{2\pi n t}{T} \right\} \to \Phi_n(t) \tag{21}$$

where (cf.Eq.(1)) Φ_n is the superoscillatory function

$$\Phi_n(t) \equiv \frac{1}{\delta_n\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}u \exp\{\mathrm{i}2\pi t \nu(u)\} \exp\left\{-\frac{1}{2\delta_n^2}(u - \mathrm{i}A_n)^2\right\} \tag{22}$$

Here the frequency function $\nu(u)$ never exceeds (for real u) its band-limited value $\nu(0) \equiv \nu_0$, and A_n and δ_n will now be determined by the requirement that Φ_n superoscillates with frequency n/T for time T.

The superoscillation frequency of $\Phi_n(t)$ is $\nu(\mathrm{i}A_n)$ (cf. Eq.(3)). Thus from Eq.(21) A_n must satisfy

$$\nu(\mathrm{i}A_n) = \frac{n}{T} \tag{23}$$

We fix δ_n by requiring that the superoscillations are maintained for time T, in the sense that the replacement of Eq.(21) remains a good approximation. For this we require the next correction to the superoscillatory exponential that $\Phi_n(t)$ represents. Expanding the saddle-point approximation to Eq.(22) (analogous to Eq.(7)) for small t, we find

$$\Phi_n(t) \approx \exp\left\{\mathrm{i}\frac{2\pi n t}{T}\right\} \exp\left\{2\pi^2\delta_n^2\left[-\nu'^2(\mathrm{i}A_n)\right]t^2\right\} \tag{24}$$

The second factor is an increasing exponential, because $\nu'(\mathrm{i}A_n)$ is imaginary, and must remain close to unity for $0<t<T$. Thus

$$\delta_n << \left[2\pi\left|\nu'(\mathrm{i}A_n)\right|T\right]^{-1} \tag{25}$$

Choosing A_n and δ_n as in Eqs.(23) and (25) guarantees that the signal $B_n(t)$, with its frequencies up to $\nu_{\max}$, will be imitated for time T. When $t>T$ the imitation will grow rapidly in strength, and eventually, that is when it is oscillating at the frequency ν_0 corresponding to its Fourier content, it will acquire an amplification factor corresponding to its largest Fourier component $n=N$. An argument analogous to that leading to Eq.(8) gives this factor as

$$F = \exp\left\{\frac{A_N^2}{2\delta_N^2}\right\} >> \exp\left\{A_N^2\pi^2T^2\left|\nu_N'(\mathrm{i}A_N)\right|^2\right\} \tag{26}$$

with A_N determined by Eq.(23) with the right-hand side set equal to $\nu_{\max}$.

Let us calculate this amplification for the model frequency function

$$\nu(u) = \nu_0 \exp\left\{-u^2\right\} \tag{27}$$

(cf. $k_3(u)$ in Eq.(2)). We find

$$A_N^2 = \log\left\{\frac{v_{\max}}{v_0}\right\} \tag{28}$$

and hence, from Eq.(26),

$$F >> \exp\left\{4\pi^2 \log^2\left(\frac{v_{\max}}{v_0}\right)v_{\max}^2 T^2\right\} \tag{29}$$

For Beethoven's ninth symphony this gives

$$F >> \exp\left\{10^{19}\right\} \tag{30}$$

This amplification will not be achieved until a time t_F, which can be estimated by the argument preceding Eq.(8) as

$$t_F \sim \left[v_0 \delta_N^2\right]^{-1} \sim \frac{v_{\max}^2 T^2}{v_0} \sim 10^8 \text{years} \tag{31}$$

Other choices for $v(u)$ give similar expressions and numerical estimates.

The estimate of Eq.(30) indicates that to reproduce music as superoscillations requires a signal with so much energy as to be hopelessly impractable, but more modest bandwidth compression might be feasible.

5. Concluding remarks

Aharonov's discovery, elaborated here, could have applications in several branches of physics. One possibility is the use of superoscillations for bandwidth compression as discussed in §4. Another example, also in signal processing, concerns the observation of oscillations faster than those expected on the basis of applied or inferred filters. These would conventionally be interpreted as high frequencies leaking through imperfect filters, but the arguments presented here show that the phenomenon could have a quite different origin, namely superoscillations compatible with perfect filtering.

Perhaps more interesting are the possible applications of superoscillatory functions of two variables, representing images. One envisages new forms of microscopy, in which structures much smaller than the wavelength λ would be resolved by representing them as superoscillations. (This is different from conventional superresolution, which is based on the fact that Fourier components larger than $2\pi/\lambda$ can be present in the field near the surface of an object, but decay exponentially away from the object because the wavenumber in the perpendicular direction is imaginary. With superoscillations, the larger Fourier components are not present.)

Superoscillations can probably exist in random functions $f(x)$: arbitrarily long intervals, in which f is exponentially small relative to elsewhere, could superoscillate. Consider how this might be achieved. If f is Gauss-distributed, its statistics are completely described by its autocorrelation function, which by the Wiener-Khinchin theorem is the Fourier transform of the power spectrum $S(q)$ of f. Even if f is band-limited, it ought to be possible to choose $S(q)$ with analytic structure (saddles with Re $q > 1$, etc.) such that the autocorrelation superoscillates as it falls from its initial value. This idea is worth pursuing.

On the purely mathematical side, it is clear that superoscillations carry a price: the function is exponentially smaller than in the regime of conventional oscillations, with the exponent increasing with the size of the interval of superoscillations. We have seen examples of this, but there ought to be a general theorem (perhaps based on a version of the uncertainty principle).

6. Acknowledgments

I thank Professors Jeeva Anandan and John Safko for arranging, and generously supporting my participation in, Yakir Aharonov's birthday meeting, and inviting me to write this paper, and Professor Ingrid Daubechies for suggestions leading to the calculation of §4.

7. Reference

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