True Quantum Chaos? An Instructive Example

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Abstract. Any chaotic classical system can be transformed into a quantum system that preserves the chaos, because the classical Liouville equation involving $2N$ phase-space variables q,p has the form of a 'Schrödinger equation' with 'coordinates' $Q=\{q,p\}$. The feature of this quantum system that allows chaos to persist is linearity of the 'Hamiltonian' in the $2N$ 'momentum' operators conjugate to Q.

It is commonly stated that there is no quantum chaos. Various reasons are given: driven quantum systems absorb energy more slowly than their chaotic classical counterparts [1], bound systems have discrete energy levels, the Schrödinger equation is linear, Planck's constant $\hbar$ smooths away classical phase-space fine structure [2-4] or replaces it, effectively, by a discrete lattice [5]. Here I want to show with a curious example that it is in fact possible to construct quantum systems with genuinely chaotic behaviour, and thereby demonstrate that some of the above statements are not true. However, the systems are so exceptional that the reasons why they allow quantum chaos to exist can help us understand why quantum chaos does not occur in situations that are physically more reasonable. The example is in essence the same as that described by Chirikov et al.[6], but I wish to present it somewhat differently and investigate it in a little more detail.

Consider a classical time-independent Hamiltonian $H_c(q,p)$ with N freedoms, generating bounded and chaotic trajectories in all or part of the $2N$-dimensional phase space. The Liouville equation governing the evolution of classical phase-space densities $\rho(q,p,t)$ is

$$\frac{\partial \rho}{\partial t} + \dot{q}\cdot\nabla_q\rho + \dot{p}\cdot\nabla_p\rho = 0 \tag{1}$$

As has often been pointed out, the fact that this equation is linear and yet preserves the chaos of the trajectories disposes of the argument that the linearity of the Schrödinger equation is responsible for the absence of chaos in quantum mechanics. Indeed, the Liouville equation becomes a 'Schrödinger equation' if ρ is regarded as a 'wavefunction' depending on $2N$ 'coordinates' Q which are simply the variables of the original phase space:

$$Q \equiv \{Q_1 \ldots Q_{2N}\} = \{q,p\} \tag{2}$$

Now let $V(Q)$ denote the original classical phase-space velocity and P the momentum operator conjugate to Q, i.e

$$V(Q) \equiv \{\dot{q},\dot{p}\} = \left\{\nabla_p H_c, -\nabla_q H_c\right\}, \quad P \equiv -i\hbar\nabla_Q \tag{3}$$

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Then on multiplication by $i\hbar$, (1) becomes

$$i\hbar\frac{\partial\rho(Q,t)}{\partial t}=\mathcal{H}_{qu}\rho(Q,t) \tag{4}$$

involving the Hermitian 'quantum' Hamiltonian operator, depending on $4N$ Q and P operators,

$$\mathcal{H}_{qu}=\frac{1}{2}\{V(Q)\cdot P+P\cdot V(Q)\} \tag{5}$$

(the expected commutator term is absent because div$V$ =0). See [14] for another problem where a 'doubled phase space' arises.

The 'Schrödinger lookalike' (4) is of course exactly the Liouville equation (1) and so has the same solutions. In particular, the contours of any evolving density develop infinite complexity (in the form of whorls and tendrils [7]) as the trajectories that convect them separate exponentially. Thus these 'wavefunctions' are chaotic: there is no $\hbar$-smoothing, no quantal suppression of chaos.

The eigenfunctions of $\mathcal{H}_{qu}$ can be chaotic too. These are of two sorts. First are uniform densities on invariant manifolds of the original classical motion (in Q space), which have 'energy' $\mathcal{E}$=0. The manifolds could be whole energy surfaces, or closed orbits. In the latter case, an eigenfunction supported by a very long closed orbit (or combination of closed orbits) can be arbitrarily complicated, because the closed orbits shadow the non-closed, chaotic orbits. Second are densities restricted to individual closed orbits, with ρ varying periodically round the orbit and proportional to

$$\exp\{in(\theta-\omega t)\} \tag{6}$$

where θ is the angle variable (equal θs in equal times) and $T=2\pi/\omega$ is the period. These are easily seen to satisfy (4) with 'energy'

$$\mathcal{E}(n,T)=n\hbar\omega=nh/T \tag{7}$$

These eigenfunctions are also chaotic for long orbits, with the additional complication when n is large that nearby parts of an orbit will have uncorrelated phases $n\theta$. The spectrum has an infinite degeneracy at $\mathcal{E}$=0, and is continuous elsewhere, because each closed orbit exists for a range of values of the original classical energies (i.e. of H_c); the set of states labelled by a given value of H_c forms a dense point spectrum. All states are localized in Q.

These wavefunctions should be regarded as the result of quantizing not the original Hamiltonian $H_c(Q)$ but the artificial classical Hamiltonian $\mathcal{H}_c(Q,P)$ corresponding to (5), namely

$$\mathcal{H}_c(Q,P)=V(Q)\cdot P \tag{8}$$

The equations of motion for the $4N$ phase-space variables are

$$\dot{Q}=V(Q),\qquad \dot{P}=-\mathbf{A}(Q(t))P,$$

$$\text{where } \mathbf{A}_{mn} \equiv \frac{\partial V_n}{\partial Q_m}, \text{ and } 1 \le m,n \le 2N \tag{9}$$

In the first of these equations, the original phase-space motion simply reappears, with all its chaos, as motion of the 'coordinates' Q, which are independent of the motion of the new 'momenta' P. The second equation shows that the P are slaved to the Q by a linear matrix equation which is exactly that governing the linearized small deviations from the orbit $Q(t)$.

Before pointing out the essential peculiarity of $\mathcal{H}_{qu}$ which enables it to possess chaos, I note some other unusual features of this Hamiltonian and its classical counterpart $\mathcal{H}_c$. First, the quantum states localized in Q on individual invariant manifolds (e.g. an energy surface, or a closed orbit) are not normalizable, because these manifolds have zero measure in Q space. Second, there are two classically conserved quantities: the original energy $H_c(Q)$, and the value of the new Hamiltonian $\mathcal{H}_c(Q,P)$. Third, in the full phase space $\{Q, P\}$ the motion is unbounded, because according to (9) the 'small deviations' P grow when the Q motion is chaotic. Fourth it follows from this that the closed orbits in Q are not periodic in Q,P. It is amusing that the 'amplification factor' for $2N$-1 of the P's can be expressed as a 'nonabelian geometric amplitude factor' [8] depending only on the geometry of the orbit's path through Q space and not on its time-dependence (the exceptional momentum corresponds to a deviation that changes the original energy).

What allows quantum chaos in this class of examples? It is the fact that the Hamiltonian $\mathcal{H}_{qu}$ is *linear in the momentum operators*. At first this seems an odd assertion, because the classical chaos, whilst not being incompatible with the fact that (8) is linear in P, is preserved when $\mathcal{H}_c$ is replaced by any function of itself (e.g. $\mathcal{H}_c{}^2$) - such transformations leave the orbits unchanged apart from an energy-dependent time rescaling. But in quantum mechanics it is the linearity in P that prevents the spreading of wavepackets which eventually obliterates chaos. This can be seen by comparing the propagators for the trivial one-dimensional quantum Hamiltonians P and P^2:

$$H = P, \quad \langle Q|\exp\{-iHt/\hbar\}|Q'\rangle = \delta(Q - Q' - t) \tag{10}$$

and

$$H = P^2, \quad \langle Q|\exp\{-iHt/\hbar\}|Q'\rangle = \frac{\exp\left\{i(Q-Q')^2/4\hbar t\right\}}{\sqrt{2i\hbar t}} \tag{11}$$

For P and P^2 the orbits are lines of constant P, but the fact, classically trivial, that the velocity of Q is independent of P in the first case, and proportional to P in the second case, has a crucial effect on the Q-propagator. (Because of linear canonical invariance, this argument about what permits quantum chaos is not restricted to Hamiltonians linear in the $2N$ Ps: it also holds for Hamiltonians that depend linearly on any commuting set of $2N$ of the Qs and Ps.)

It is worth remarking that the P-linearity of $\mathcal{H}_{qu}$, and hence the associated quantum chaos, is not destroyed by arbitrary canonical transformations among the original phase-space variables Q. In the doubled phase space these are just coordinate transformations, and generate new Ps depending linearly on the old ones.

Linearity permits quantum chaos but of course does not guarantee it. Obviously there is no chaos in (10), which was intended only to illustrate nonspreading. And if

in the classically chaotic and quantally nonchaotic kicked rotator [1,5] the kinetic energy term P^2 is replaced by P, the aim is achieved of making the quantum motion exactly follow the classical motion [9], but at the price of destroying the chaos. Another class of classically chaotic but quantally nonchaotic linear Hamiltonians are those which, as in neutrino billiards [10], act on spinors rather than scalars, - although Pasmanter [11] gives two interesting examples where quantum spinorization does not destroy the chaos.

It is curious that whereas in classical mechanics chaos requires *nonlinearity* in the equations of motion (that is, $H_c(Q)$ must be more than quadratic in some of its variables), the preservation of that chaos into the 'Schrödinger lookalike' via Liouville's equation requires *linearity* of $\mathcal{H}_{qu}$ in P.

When the quantum Hamiltonian is not linear in momentum, we have the now-familiar situation: quantum mechanics suppresses the chaos, and the characteristic phenomena of quantum chaology appear [12,13].

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Toward the Fundamental Theory of Nuclear Matter Physics: The Microscopic Theory of Nuclear Collective Dynamics

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Abstract. Since the research field of nuclear physics is expanding rapidly, it is becoming more imperative to develop the microscopic theory of nuclear matter physics which provides us with a unified understanding of diverse phenomena exhibited by nuclei. An establishment of various stable mean-fields in nuclei allows us to develop the microscopic theory of nuclear collective dynamics within the mean-field approximation. The classical-level theory of nuclear collective dynamics is developed by exploiting the symplectic structure of the time-dependent Hartree-Fock (TDHF)-manifold. The importance of exploring the single-particle dynamics, e.g. the level-crossing dynamics in connection with the classical order-to-chaos transition mechanism is pointed out. Since the classical-level theory is directly related to the full quantum mechanical boson expansion theory via the symplectic structure of the TDHF-manifold, the quantum theory of nuclear collective dynamics is developed at the dictation of what is developed in the classical-level theory. The quantum theory thus formulated enables us to introduce the quantum integrability and quantum chaoticity for individual eigenstates. The inter-relationship between the classical-level and quantum theories of nuclear collective dynamics might play a decisive role in developing the quantum theory of many-body problems.

1. Introduction

1.1. Basic Problems in Nuclear Matter Physics

In the light of the recent developments of nuclear physics, new trends of nuclear collective dynamics in the 1990s seem to be focussing on the following three subjects; first: the dynamics of symmetry breaking, i.e. the dynamical relation among many vacua with different symmetries, second: the dynamics of the order-to-chaos transition mechanism in the quantum system, and: the dynamics of the many-body quantum system in terms of geometrical or topological structure of the symplectic manifold. Nowadays, these basic problems are becoming rather common in many other fields of theoretical physics. It is an objective of the present contribution to try to explain how suitable it is to exam-

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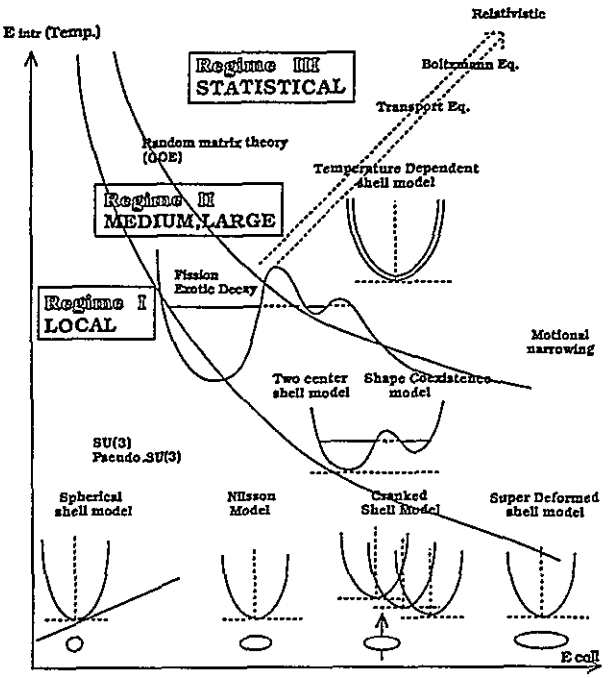

Figure 1. Three characteristic regimes of nuclear matter physics.

ine these fascinating subjects in the field of nuclear physics. With this aim, let us start by summarizing the present status of nuclear physics.

With the aid of recent improved experimental facilities both in accelerators and in complex detector systems, a great variety of phenomena are being extensively studied, ranging from the low-lying collective excited states up to the violent high energy heavy-ion reactions in the so-called BUU or VUU regime [1]. These phenomena might be classified into three characteristic regimes shown in the phase diagram in Fig. 1. The first regime is characterized by ***local phenomena*** which are understood by introducing an appropriate stable mean-field with one local minimum specified by a certain symmetry, special coupling scheme, group theoretical model space or phenomenological collective subspace etc. In the second regime, there are ***medium- or large-amplitude collective phenomena*** which are usually described by a suitable mean-field with many local minima with different symmetries. The third regime consists of ***statistical or probabilistic phenomena*** which are explained by means of thermo-dynamical concepts like the transport equation, stochastic equation, temperature, dissipation, equation of states etc. Since a great store of knowledge on individual phenomena is now available and since our knowledge on each phenomenon is expected to expand rapidly, it becomes quite imperative to develop the fundamental theory of nuclear matter physics which provides us