

## A rule for quantizing chaos?

M V Berry and J P Keating

H H Wills Physics Laboratory, Tyndall Avenue, Bristol BS8 1TL, UK

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**Abstract.** We find a real function  $\Delta(E)$  whose zeros approximate the quantum energy levels of a system with chaotic classical trajectories.  $\Delta(E)$  is a finite sum over combinations of classical periodic orbits. It is obtained from Gutzwiller's infinite and divergent sum, representing the spectral density in terms of periodic orbits, by means of a resummation conjectured by analogy with a derivation of the Riemann-Siegel formula for the Riemann zeros. We assess the practicality of the quantization condition.

### 1. Introduction

We seek an approximate quantization formula for the energy levels  $E_j$  of a bound quantum system whose Hamiltonian  $\hat{H}$  has a classical limit with chaotic trajectories. Despite many attempts (reviewed by Berry 1983 and Eckhardt 1988) such a formula has remained elusive, the main difficulty being the correct incorporation into quantum mechanics of the influence of the long classical orbits (Berry 1985, 1990)—a problem that does not arise for integrable systems. Here we intend to tackle this difficulty head on and obtain a rule

$$\Delta(E) = 0 \tag{1}$$

determining the  $E_j$  as the zeros of a real function  $\Delta$  containing explicit information about the classical motion involving only orbits of finite length.

The hope is that the energies generated by (1) will be semiclassical approximations, with errors which vanish faster than the mean level separation as Planck's constant  $\hbar \rightarrow 0$ . A semiclassical approximation which fails this test is

$$\Delta(E) \approx \cos\{\pi\bar{\mathcal{N}}(E)\} \tag{2}$$

in which

$$\bar{\mathcal{N}}(E) = \frac{1}{h^D} \int d^D q d^D p \Theta\{E - H(\mathbf{q}, \mathbf{p})\} \tag{3}$$

is the smoothed (Weyl) spectral staircase (counting function), giving the average number of levels  $E_j < E$ .  $D$  is the number of freedoms of the classical system, with Hamiltonian  $H(\mathbf{q}, \mathbf{p})$ , and  $\Theta$  denotes the unit step. The formula (2) is too crude because it ignores the spectral fluctuations whose nature (Bohigas and Giannoni 1984, Berry 1987) reflects the underlying classical chaos. Nevertheless, the zeros of (2), where  $\bar{\mathcal{N}} = (j + \frac{1}{2})$ , do form a sequence of energies with the correct density, and will be the starting point for our treatment.

The derivation of the function  $\Delta(E)$  will be based on a conjectured resummation of the divergent tail of the formula obtained by Gutzwiller (1971) (see also Littlejohn 1990) for the spectral density as a sum over periodic classical trajectories. The resummation is motivated by two observations. First, in order for the spectral density as represented by Gutzwiller's series to have the correct singularities, the long orbits must encode the energy levels and hence, through the long-range correlations of these levels, the short orbits; an application of this 'bootstrap' idea was given by Berry (1985). Second, there is an analogy with the Riemann-Siegel formula for the Riemann zeta function  $\zeta(z)$ . One of us (Berry 1986) has already presented an approximate version of this idea, but now we are able to work it out in more detail. Specifically, we have obtained the analogue of the well known Dirichlet series representing  $\zeta(z)$  as a sum over the integers. It is this new result which forms the basis for the proposed resummation scheme.

The argument by analogy from quantum mechanics to the Riemann zeros has proved fruitful, leading for example to accurate predictions (Berry 1988) of the statistics of its extremely high zeros. Here we argue in the reverse direction, from number theory to quantum mechanics. As we will explain, the 'Riemann-Siegel lookalike' formula has several attractive features, but might not be computationally effective.

When this paper was in a late stage of preparation we received a preprint from Bogomolny (1990), describing a quantization condition in the form of an infinite complex determinant generated by a Poincaré section of the classical motion. At this stage we are not sure how Bogomolny's determinant can be expanded to yield our 'Riemann-Siegel' expansion, or how our expansion can be compactified to give his determinant.

**2. Spectral determinant**

We start with an expression which obviously has zeros at the eigenvalues  $E_j$ , namely

$$\Delta(E) \equiv \det\{A(E, \hat{H})(E - \hat{H})\} = \prod_j \{A(E, E_j)(E - E_j)\} \tag{4}$$

where  $A$ , which has no real zeros, is introduced to make the product converge. Voros (1987) has given a thorough discussion of such regularizations. Typical examples are  $A = -1/E_j$ , which works for the particle in a one-dimensional box and leads to

$$\Delta(E) = \prod_{j=1}^{\infty} (1 - E/j^2) = \frac{\sin(\pi\sqrt{E})}{\pi\sqrt{E}} \tag{5}$$

and  $A = -(1/E_j) \exp\{E/(E_j + 1/2)\}$ , which works for the harmonic oscillator and leads to

$$\begin{aligned} \Delta(E) &= \prod_{j=1}^{\infty} [1 - E/(j + 1/2)] \exp\{E/(j + 1)\} \\ &= \frac{\exp(\gamma E)\Gamma(E + 1/2) \sin\{\pi(E + 1/2)\}}{\sqrt{\pi}} \end{aligned} \tag{6}$$

where  $\gamma$  is Euler's constant. The precise form of the regularizer  $A$  will not be important here.

We shall need the trace of the resolvent, defined for arbitrary complex  $E$  by

$$g(E) \equiv \text{Tr} \frac{1}{E - \hat{H}} \tag{7}$$

(regularized if necessary). In terms of  $g$ , the spectral staircase is

$$\begin{aligned} \mathcal{N}(E) &= -\frac{1}{\pi} \text{Im} \text{Tr} \log\{1 - (E + i\varepsilon)/\hat{H}\} \\ &= -\frac{1}{\pi} \text{Im} \int_0^E dE' g(E' + i\varepsilon) \end{aligned} \tag{8}$$

where  $E$  is real. The connection with  $\Delta(E)$  comes from

$$\Delta(E) = \det A \exp \text{Tr} \log(E - \hat{H}) \tag{9}$$

which can be written in the form

$$\Delta(E) = B(E) \exp \left\{ -i\pi\bar{\mathcal{N}}(E) + \int_0^E dE' [g(E') - \bar{g}(E')] \right\} \tag{10}$$

where  $B(E)$  is real and non-zero for real  $E$  and  $\bar{g}$  is the smoothed resolvent trace.

We shall use Gutzwiller's formula for  $g - \bar{g}$  as a sum over the classical periodic orbits with energy  $E$ ; because the system is chaotic it is assumed that these orbits are isolated and unstable. If  $p$  labels the primitive orbits and  $m$  labels their repetitions, then in the semiclassical limit (Gutzwiller 1971)

$$g(E) - \bar{g}(E) \approx \frac{-i}{\hbar} \sum_p \sum_{m=1}^{\infty} \frac{T_p \exp\{imS_p/\hbar\}}{\sqrt{|\det(\mathbf{M}_p^m - 1)|}} \tag{11}$$

Here  $S_p$  is the action of the primitive orbit (in which for convenience we have incorporated the Maslov index),  $T_p = dS_p/dE$  is its period, and  $\mathbf{M}_p$  its linearized Poincaré map; all three quantities depend on  $E$ , but we have not indicated this explicitly. Thus (10) becomes

$$\Delta(E) = B(E) \exp\{-i\pi\bar{\mathcal{N}}(E)\} \prod_p \exp \left\{ -\sum_{m=1}^{\infty} \frac{\exp\{imS_p/\hbar\}}{m\sqrt{|\det(\mathbf{M}_p^m - 1)|}} \right\}. \tag{12}$$

The difficulty is that, because of the exponential proliferation of long orbits (as  $\exp(\lambda(E)T)/\lambda(E)T$ , where  $\lambda$  denotes the topological entropy), (11) and (12) diverge when  $E$  is real. These sums diverge even when, as for quantum billiards on the pseudosphere (Balazs and Voros 1986), they are exact for complex  $E$ , in which case they constitute a version of the Selberg trace formula. To make the sums converge, we would need

$$\text{Im } E > \frac{1}{2}\lambda(E)\hbar \tag{13}$$

(see, e.g., Eckhardt and Aurell 1989 or Berry 1990, and the classical sum rule of Hannay and Ozorio de Almeida 1984). However, we are forced across this 'entropy barrier', because we seek to discriminate individual levels and so must keep  $E$  real. To motivate the manipulations inspired by this observation, we must briefly review the analogous situation for the Riemann zeta function.

### 3. Riemann–Siegel analogy

Riemann’s zeta function is defined as

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \tag{14}$$

in the region  $\text{Re } z > 1$  where the sum converges, and by analytic continuation elsewhere. ‘Elsewhere’ includes the critical line  $\text{Re } z = \frac{1}{2}$  where according to the Riemann hypothesis the non-trivial zeros lie. We seek an asymptotic ‘quantization condition’ determining the imaginary parts  $E$  of the zeros that are far from the real axis.

An exact condition for the zeros is

$$\Delta(E) \equiv \exp\{-i[\text{Im } \log \Gamma(\frac{1}{4} + \frac{1}{2}iE) - \frac{1}{2}E \log \pi]\} \zeta(\frac{1}{2} - iE) = 0. \tag{15}$$

It follows from the functional equation for  $\zeta(z)$  that  $\Delta(E)$  is real for real  $E$  (Edwards 1974). The phase in the exponential is related to the smoothed spectral staircase for the zeros:

$$\text{Im } \log \Gamma(\frac{1}{4} + \frac{1}{2}iE) - \frac{1}{2}E \log \pi = \pi[\bar{\mathcal{N}}(E) - 1] \rightarrow \frac{1}{2}E \log \frac{E}{2\pi e} - \frac{\pi}{8} \quad \text{as } E \rightarrow \infty. \tag{16}$$

If we naively employ the Dirichlet series (14) on the critical line, we obtain

$$\Delta(E) \sim -\exp[-i\pi\bar{\mathcal{N}}(E)] \sum_{n=1}^{\infty} \frac{\exp(iE \log n)}{n^{1/2}} \tag{17}$$

an expression that is neither real nor convergent. This difficulty can be overcome by truncating the series (17) and resumming its divergent tail by the Poisson formula together with the method of stationary phase (Berry 1986). The remarkable result is that for large  $E$  and truncation at  $n = \text{Int}[\sqrt{(E/2\pi)}]$  the resummed tail is itself a series, now finite, reproducing term by term the original series, with the crucial difference that the tail terms are complex conjugates of the ‘head’ terms. Incorporating this feature of self-reproduction, or ‘resurgence’, or bootstrapping, we obtain the real expression

$$\Delta(E) \approx -2 \sum_{n=1}^{\text{Int}[\sqrt{E/2\pi}]} \frac{\cos\{\pi\bar{\mathcal{N}}(E) - E \log n\}}{n^{1/2}}. \tag{18}$$

This finite sum is the Riemann–Siegel formula (Edwards 1974) which (with several correction terms) is employed in effective methods for computing Riemann zeros up to—at least—the 10<sup>20</sup>th (Odlyzko 1990).

The leading term  $n = 1$  in (18) is just the crude approximation (2), oscillating on the scale of the mean separation of the zeros. Higher terms oscillate more slowly, and the highest term, at the truncation limit  $n = \text{Int}[\sqrt{(E/2\pi)}]$ , is locally stationary in  $E$ . This follows from

$$\frac{d}{dE} \{\pi\bar{\mathcal{N}}(E) - E \log n\} = \log \left\{ \frac{1}{n} \sqrt{\frac{E}{2\pi}} \right\} = 0. \tag{19}$$

The Riemann–Siegel formula has a ‘semiclassical’ interpretation, following from a similar interpretation (Berry 1985, 1990) for  $\log \zeta$  when expanded as an Euler product. According to this, there is a classical system underlying the zeros, whose primitive periodic orbits have periods  $T = \log p$  where  $p$  is prime; therefore repeated orbits have

periods  $m \log p$  where  $m$  is integer. If the term  $E \log n$  in (18) is interpreted as an action, the corresponding period is

$$\log n = \log \prod_p p^{m_p} = \sum_p m_p \log p \tag{20}$$

where  $p$  and  $m_p$  are the primes and powers in the decomposition of  $n$ . Thus (18) is a sum over all linear combinations of orbits and repetitions, that is over ‘pseudo orbits’ (Berry 1986). This picture of  $\Delta(E)$  as a real sum over pseudo-orbits, truncated at the term which does not oscillate, is what we now generalize.

#### 4. Riemann–Siegel lookalike

We will manipulate (12) into an analogue of the Dirichlet series (17), as it is from this representation that the Riemann–Siegel formula was derived. Restricting ourselves to  $D = 2$ , and remembering that we are considering unstable systems, we write the two eigenvalues of the symplectic Poincaré map  $M_p$  as

$$\begin{aligned} \exp(\pm \lambda_p T_p) & \quad (\text{map hyperbolic}) \\ -\exp(\pm \lambda_p T_p) & \quad (\text{map hyperbolic with reflection}) \end{aligned} \tag{21}$$

where  $\lambda_p$  is the instability exponent of the primitive orbit  $p$ ; for long orbits,  $\lambda_p$  tends to the Kolmogorov entropy of the system. For simplicity we give the argument only for the first case, and later state the modification necessary when the map is hyperbolic with reflection. Thus

$$|\det(M_p^m - 1)| = \exp(m\lambda_p T_p) [1 - \exp(-m\lambda_p T_p)]^2. \tag{22}$$

In (12) we have

$$\frac{1}{\sqrt{|\det(M_p^m - 1)|}} = \exp(-\frac{1}{2}m\lambda_p T_p) \sum_{k=0}^{\infty} \exp(-mk\lambda_p T_p). \tag{23}$$

Now the sum over  $m$  in (12) can be evaluated as a logarithm, and we obtain

$$\Delta(E) = B(E) \exp\{-i\pi\bar{N}(E)\} \prod_p \prod_{k=0}^{\infty} [1 - \exp\{-(k + \frac{1}{2})\lambda_p T_p\} \exp\{iS_p/\hbar\}]. \tag{24}$$

This is a dynamical zeta function, the most familiar example being Selberg’s zeta function (McKean 1972). To obtain a sum rather than a product, we first make use of Euler’s identity

$$\prod_{k=0}^{\infty} (1 - ax^k) = \sum_{m=0}^{\infty} \frac{a^m (-1)^m x^{m(m-3)/4}}{(x^{-1/2} - x^{1/2})(x^{-1} - x) \dots (x^{-m/2} - x^{m/2})}. \tag{25}$$

Thus

$$\begin{aligned} \Delta(E) &= B(E) \exp\{-i\pi\bar{N}(E)\} \\ &\times \prod_p \left[ \sum_{m=0}^{\infty} (-1)^m \exp\{-\frac{1}{4}m(m-1)\lambda_p T_p\} \exp\{imS_p/\hbar\} \right. \\ &\times \left. \left( \prod_{j=1}^m \det(M_p^j - 1) \right)^{-1/2} \right] \end{aligned} \tag{26}$$

where the term  $m = 0$  is unity.

The phase factors suggest interpreting the new summation variable  $m$  as a repetition number. Expanding the product over primitive orbits gives the desired series, analogous to (17), which is the central result of this analysis:

$$\Delta(E) = B(E) \exp\{-i\pi\bar{N}(E)\} \sum_{n=0}^{\infty} C_n(E) \exp\{i\mathcal{S}_n(E)/\hbar\}. \tag{27}$$

Here  $n$  labels pseudo orbits, in which each primitive orbit  $p$  is repeated  $m_p$  times (including zero), i.e.

$$n = \{m_p\}. \tag{28}$$

We choose the labelling such that increasing  $n$  corresponds to increasing period of the pseudo orbits.  $\mathcal{S}_n$  denotes the action of the pseudo-orbit:

$$\mathcal{S}_n \equiv \sum_p m_p \mathcal{S}_p. \tag{29}$$

The coefficient  $C_n$  is

$$C_n \equiv \prod_p \left[ (-1)^{m_p} \exp\{-\frac{1}{4}m_p(m_p - 1)\lambda_p T_p\} \left( \left| \prod_{j=1}^{m_p} \det(\mathbf{M}_p^j - 1) \right| \right)^{-1/2} \right]. \tag{30}$$

In the product, primitive orbits whose  $m_p = 0$  give factors unity. For primitive orbits that are hyperbolic with reflection, the contribution is identical in form, except for the replacement

$$(-1)^{m_p} \rightarrow (-1)^{\text{Int}[(m_p + 1)/2]}. \tag{31}$$

If we try to imitate for  $\zeta(z)$  the analysis leading from (12) to (27), that is to obtain the Dirichlet series from the exponential of the expansion of the logarithm of the Euler product, we encounter two differences at the start. Instead of (12), we have

$$\begin{aligned} \zeta(\frac{1}{2} - iE) &= \exp \log \prod_p \left( 1 - \frac{\exp\{iE \log p\}}{p^{1/2}} \right)^{-1} \\ &= \prod_p \exp \left\{ + \sum_{m=1}^{\infty} \frac{\exp(imE \log p)}{mp^{m/2}} \right\}. \end{aligned} \tag{32}$$

The first difference is that here the exponent has a + sign instead of the - in (12); it is as though all the ‘Riemann orbits’—including repetitions—possess a Maslov index of  $\pi$ . The second difference is that instead of the determinant in the denominator of (12) there is here simply a power. What is peculiar is that these differences cancel, as it were: the absence of a determinant means that the ‘repetition’ sum can be evaluated immediately as a logarithm, bypassing the need for the intermediate sum over  $k$  as in (23), but the logarithm appears with the ‘wrong’ sign, leading instead of (24) to a product of reciprocals, whose expansion reinstates the repetitions in the pseudo-orbits.

An interesting feature of (30) is the rapid decrease in the contribution to pseudo-orbits from multiple repetitions: for large  $m_p$  (or large instability exponent  $\lambda_p$ ), we find

$$\exp\{-\frac{1}{4}m_p(m_p - 1)\lambda_p T_p\} \left( \left| \prod_{j=1}^{m_p} \det(\mathbf{M}_p^j - 1) \right| \right)^{-1/2} \rightarrow \exp\{-\frac{1}{2}m_p^2 \lambda_p T_p\}. \tag{33}$$

Thus the most important long pseudo-orbits are those composed of primitive orbits traversed once.

So far our manipulations have been purely formal, and the resulting series (27) for the spectral determinant shares the disadvantages of the periodic-orbit sum (11) from which it was obtained: for real  $E$  it diverges, and any truncation of it is complex rather than real. The final step is to argue, by analogy with the Riemann–Siegel formula, that the appropriate resummation of the tail of the series reproduces, one by one, the complex conjugates of the terms in the head of the series. If this is correct, (27) can be rewritten to give our main result: in the semiclassical limit, energy levels are given by the finite quantum condition

$$\Delta(E) = 2B(E) \sum_{n=0}^{\mathcal{T}_n < \mathcal{T}^*(E)} C_n(E) \cos\{\mathcal{S}_n(E)/\hbar - \pi\bar{\mathcal{N}}(E)\} = 0 \tag{34}$$

over pseudo-orbits with periods  $\mathcal{T}_n$  (defined by analogy with the actions (29)) less than the value  $\mathcal{T}^*(E)$  for which the argument of the cosine is locally stationary. This is given by

$$\frac{d}{dE} [\mathcal{S}_n(E)/\hbar - \pi\bar{\mathcal{N}}(E)] = 0 \tag{35}$$

from which we find

$$\mathcal{T}^*(E) = \frac{\hbar \bar{d}(E)}{2} \tag{36}$$

where  $\bar{d}(E)$  is the smoothed level density (derivative of (3)). It is striking—and surely not accidental—that this truncation is just what would have been expected on the basis of the uncertainty principle, ensuring that bootstrapping relates long orbits, which generate the short-range spectral correlations, to short orbits, which generate long-range correlations, the symmetry point being correctly located at the mean level spacing.

There are two reasons why the formula (34) is an attractive candidate for the correct analytic continuation of (27) into the strip where it diverges. First, it is real when  $E$  is real, as  $\Delta(E)$  must be. Second, when  $\text{Im } E > 0$  the additional terms introduced by resurgence are all exponentially smaller than the original terms, which survive into the domain where (27) converges and so match smoothly onto (27).

### 5. Discussion

We have derived for the spectral determinant  $\Delta(E)$  an analogue of the Dirichlet series representation of the Riemann zeta function. A natural resummation of this series, retaining two important analytical properties of the function, was then proposed by analogy with the Riemann–Siegel formula. We have to admit, however, that the resurgence we are conjecturing, in which the ‘tail’ terms of (27), i.e. the long orbits, regenerate the ‘head’ terms, i.e. the short orbits, would appear to require a miraculous conspiracy. A direct demonstration would require far more refined information about the actions of the long periodic orbits than we possess. We are not even able to establish the resurgence of the first term, and thereby derive by resummation the elementary result (2), equivalent to what in previous work (Berry 1985) was called the semiclassical sum rule (for a derivation of this rule in the Riemann case, see Keating 1991).

Nevertheless, some such bootstrap identity must exist if there is to be any meaningful quantization condition in which individual periodic orbits play a role. And the regeneration of the first term by the resummation of the tail occurs even in one dimension,

as we now show. For an anharmonic oscillator there is a single primitive orbit whose action  $S$  is the phase-space area of the energy contour, related to the smoothed spectral staircase by

$$\bar{\mathcal{N}} = \frac{S}{2\pi\hbar}. \quad (37)$$

The Maslov index is  $\pi$  ( $\pi/2$  from each turning point) so that in (12) the product contains one term and

$$\frac{S_p}{\hbar} = \frac{S}{\hbar} + \pi. \quad (38)$$

The determinant factor is absent in one dimension, so

$$\begin{aligned} \Delta(E) &= B(E) \exp\{-i\pi\bar{\mathcal{N}}(E)\} \exp\left\{-\sum_{m=1}^{\infty} \frac{(-1)^m \exp\{imS/\hbar\}}{m}\right\} \\ &= B(E) \exp\{-i\pi\bar{\mathcal{N}}(E)\} \exp \log[1 + \exp\{iS/\hbar\}] \\ &= B(E) \exp\{-i\pi\bar{\mathcal{N}}(E)\} [1 + \exp\{i2\pi\bar{\mathcal{N}}(E)\}] \end{aligned} \quad (39)$$

giving the quantization condition (2), which is the correct semiclassical result in one dimension.

If the quantum condition (34) is correct, it represents a considerable improvement over the unresummed Gutzwiller formula (11), employed in the spectral staircase (8) (or its derivative, the spectral density).

Without resummation, the best hope is to see the levels emerging in the form of singularities (step or delta) as more orbits are included in the sum. This will not happen if the series is evaluated for real  $E$ , because it diverges. If it is evaluated on the boundary of convergence, given by (13), the singularities can at best be resonances of width  $\text{Im } E$ , and individual levels will not be resolved if  $\text{Im } E$  exceeds the mean level separation. For example, in planar quantum billiards with area  $\mathcal{A}$  and perimeter  $\mathcal{L}$  this leads to the expectation that only the first  $\mathcal{N}_{\max}$  levels could be resolved, where

$$\mathcal{N}_{\max} = 4n_b^2 \pi^3 \frac{\mathcal{A}}{\mathcal{L}^2} \sim n_b^2 \pi^2. \quad (40)$$

Here  $n_b$  is the 'chaos bounce number', defined as the chaos time'  $1/\lambda(E)$  multiplied by the particle speed and divided by the mean distance  $\pi\mathcal{A}/\mathcal{L}$  between bounces (i.e. the number of  $n$ -bounce periodic orbits proliferates as  $\exp(n/n_b)/n$ ). And for the Riemann zeros

$$\mathcal{N}_{\max} = (4\pi - 1) \exp(4\pi) \sim 3 \times 10^6. \quad (41)$$

These limitations do not prevent the determination of low levels without resummation. This has been achieved by Gutzwiller (1971) for the anisotropic Kepler problem, and by Berry (1985) for the Riemann zeros. (Gutzwiller (1982) obtained more levels for the anisotropic Kepler problem using a resummation based on the symbolic dynamics of that particular problem.) It should also be remarked that alternative regularizations than making  $E$  imaginary, for example Gaussian smoothing (Delsarte 1966 and Aurich *et al* 1988, see also the discussion by Berry 1990) hold out the hope of generating individual high levels without resummation, but still involve infinite series and the identification of singularities.



With resummation, however, it is necessary only to find the zeros of a finite real function, which is possible in principle for arbitrarily high levels, i.e. far beyond the entropy barrier (13) underlying (40) and (41).

To assess the practicality of our procedure based on (34), we estimate the number  $N_{\text{pseud}}(\mathcal{N})$  of pseudo-orbits necessary to obtain  $\mathcal{N}$  levels. This is the number of pseudo-orbits with period less than  $\mathcal{T}^*$ , given by (36). Like ordinary orbits, pseudo-orbits proliferate exponentially with period, the exponent being  $\lambda\mathcal{T}$ , but there are more of them, and we conjecture the law

$$N_{\text{pseud}} \rightarrow \exp(\lambda\mathcal{T}) \quad \text{as} \quad \mathcal{T} \rightarrow \infty \tag{42}$$

(this is exact for the Riemann case, for which the period of the  $n$ th pseudo-orbit is  $\log n$ ). Thus from (36) we obtain

$$N_{\text{pseud}}(\mathcal{N}) = \exp\{\pi\lambda\hbar\bar{d}(E)\}. \tag{43}$$

We work this out for two examples.

For the Riemann zeros,  $\lambda = \hbar = 1$ , and the density of zeros follows from (16), giving

$$\begin{aligned} N_{\text{pseud}}(\mathcal{N}) &= \exp\left\{\frac{1}{2}\log\left(\frac{E}{2\pi}\right)\right\} = \sqrt{\left(\frac{E}{2\pi}\right)} \\ &= \sqrt{\mathcal{N}/\log(\mathcal{N}/\log(\mathcal{N}/\log\dots))} \sim \sqrt{\mathcal{N}}. \end{aligned} \tag{44}$$

Thus  $N_{\text{pseud}}$  increases more slowly than  $\mathcal{N}$ . This implies that the discontinuities in the Riemann-Siegel formula (18), caused by the jumps in its upper limit, get increasingly sparse in comparison with the zeros. This is one reason why the formula is so useful in practice. (We should remark that the Riemann-Siegel formula contains corrections (Edwards 1974), which we have not discussed, whose effect is to remove the discontinuities in successive derivatives of  $\Delta(E)$ ).

For planar billiards, as introduced before (40), a similar argument leads to

$$N_{\text{pseud}}(\mathcal{N}) = \exp\left\{\frac{\mathcal{L}}{n_b} \sqrt{\frac{\mathcal{N}}{\pi\mathcal{A}}}\right\}. \tag{45}$$

For a non-relativistic particle moving in any  $D$ -dimensional scaling potential, we find a similar expression with exponent proportional to  $\mathcal{N}^{(D-1)/D}$ . Now  $N_{\text{pseud}}$  increases faster than  $\mathcal{N}$ . This means that the work required to calculate  $\mathcal{N}$  eigenvalues increases faster than  $\mathcal{N}$ , thereby compromising the computational value of the quantum condition. If the classical orbits have a sufficiently simple symbolic organization, it might be possible to regroup the pseudo-orbits as in the curvature expansions of Cvitanović and Eckhardt (1989) and thereby substantially reduce the computational labour (this technique seems to fail for the Riemann zeros).

The rapid growth of  $N_{\text{pseud}}(\mathcal{N})$  also indicates that the discontinuities in the semi-classical spectral determinant become denser in comparison with the level spacing. However, the total contribution from the terms which appear in the range  $E_j \leq E \leq E_{j+1}$ , compared with those already present in the sum, will not exceed  $d(\log N_{\text{pseud}})/d\mathcal{N}$  which does vanish as  $\mathcal{N} \rightarrow \infty$ , and moreover the discontinuities are exponentially small; therefore we do not expect their proliferation to destroy the discrimination of the zeros. (We have not found any systematic method for removing the discontinuities.)

If these difficulties can be resolved, there arises the question of the relation between the exact energy levels and the zeros of the Riemann-Siegel lookalike approximation

for  $\Delta(E)$ . Although the approximate levels have the correct density and, one may hope, the correct fluctuation statistics, they are unlikely to correspond one by one with the exact levels. This is because we started with Gutzwiller's series (11), which is a semiclassical approximation whose corrections are of order  $\hbar^2$  and hence would shift the levels by an amount which is not small compared to their spacings. (Another way to see this is to note that our  $\Delta(E)$  contains only canonically invariant information about the periodic orbits, whereas the levels depend to order  $\hbar^2$  on the operator-ordering convention employed to quantize the system (Wilkinson 1988)).

In any case, it would be helpful to have a careful numerical exploration of the quantization formula (37), to assess the importance of the proliferating discontinuities just discussed, and discover whether levels can be determined beyond the number  $\mathcal{N}_{\max}$  (cf (40)) resolvable without resummation, determined by the entropy barrier. One candidate system is billiards on the pseudosphere (Balazs and Voros 1986). Dr M Sieber has kindly carried out a preliminary test of (37) for the planar hyperbola billiard, for which many periodic orbits are available (Sieber and Steiner 1990). This has infinite area, so that (45) does not apply. The test shows that at least the lowest ten levels can be discriminated.

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