# COMMENT ON "NEW REPRESENTATION OF QUANTUM CHAOS"

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A claim that "the irregular sequence of energy levels gives an unambiguous representation of chaos in quantum systems" is shown to be unfounded.

In a recent paper [1], the energy levels $\{E_n\} = E_1, E_2 ...$ were calculated for two quantum systems, one with an integrable classical analogue and one with a nonintegrable classical analogue. Graphs of level spacings $\Delta E_n = E_{n+1} - E_n$ versus $n$, and level spacing correlation plots of $\Delta E_{n+1}$ versus $\Delta E_n$, were presented and adduced to support the claim quoted in the abstract. To refute this claim it is sufficient to present as counterexample a classically integrable system with an irregular level structure. This is easy to do because, as was shown some years ago [2], the quantum levels for *almost all* classically integrable systems – the most important exceptions being coupled harmonic oscillators – form locally uncorrelated sequences of random numbers, as $n \to \infty$. This implies not only that high level spacings have an exponential distribution but also that successive spacings are uncorrelated.

An integrable system for which random level structure can be seen even in low-lying levels is the *particle in a two-dimensional rectangular box* with ratio $\alpha$ of side lengths (where $\alpha^2$ is irrational), for which the levels, normalized to have unit mean density, are given in terms of quantum numbers $m$, $n$ by

$$E(m, n) = \tfrac{1}{4}\pi(m^2/\alpha + n^2\alpha) \quad (1 \leqslant m, n < \infty). \qquad (1)$$

Fig. 1a shows the spacings graph of the first 100 levels, for $\alpha = \pi$. This should be compared with fig. 1b which shows spacings drawn at random from an exponential distribution with unit mean. Figs. 2a and 2b show the corresponding correlation plots. The obvious irregularity of the sequence calculated from (1) refutes the claim made in ref. [1].

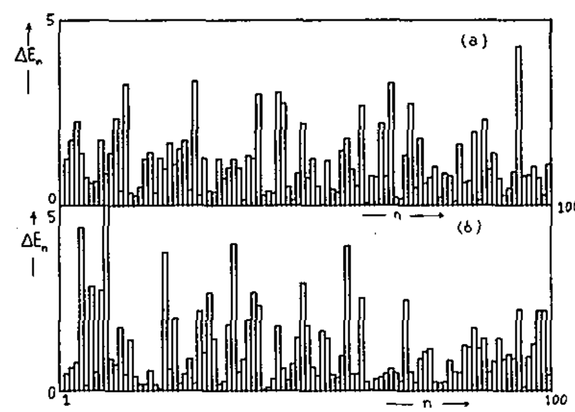


Fig. 1. (a) Spacings graph of $\Delta E_n = E_{n+1} - E_n$ versus $n$ for $1 \leqslant n \leqslant 100$, for particle in a rectangular box with side ratio $\pi$; (b) 100 uncorrelated random spacings, exponentially distributed.

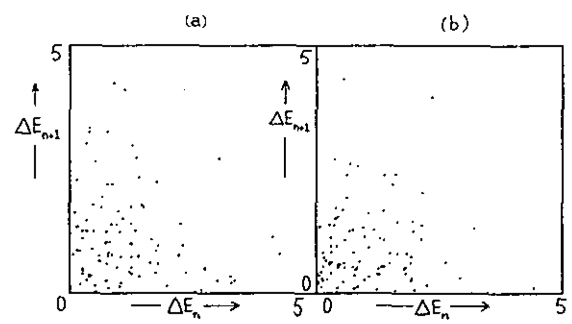


Fig. 2. (a) Correlation plot of $\Delta E_{n+1}$ versus $\Delta E_n$, corresponding to the "box" spacings in fig. 1a; (b) Correlation plot corresponding to the random spacings in fig. 1b.



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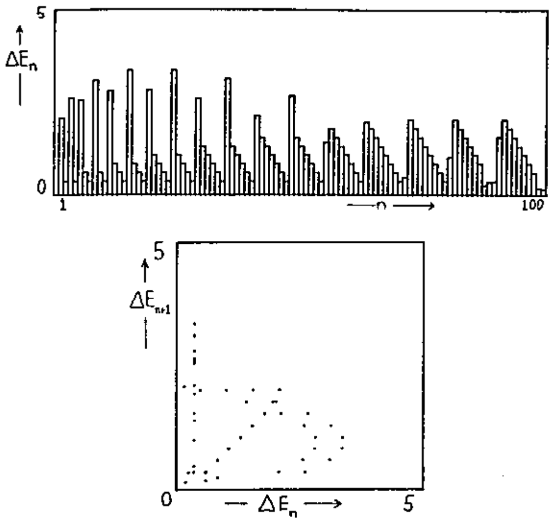

Fig. 3. Top: spacings graph for $1 \leqslant n \leqslant 100$ for perturbed oscillator with levels given by eqs. (3) and (4) with $\alpha = 1/2$, $\delta = 0.11012$. Bottom: correlation plot.

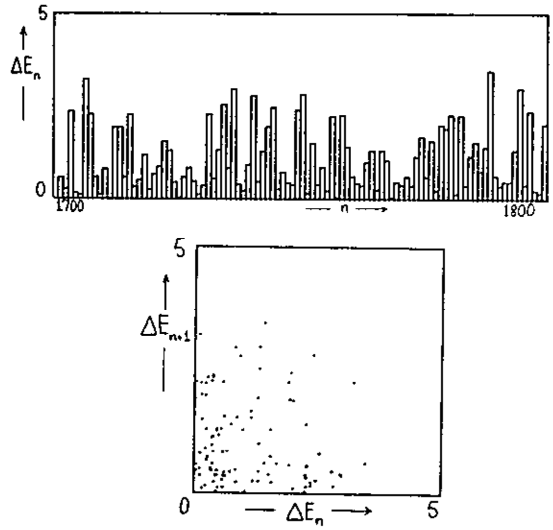

Fig. 4. Top: spacings graph, as in fig. 3, but for $1700 \leqslant n \leqslant 1800$. Bottom: correlation plot.

For some integrable systems, the geometry of the function $E(m, n)$ on the lattice of quantum numbers $m, n$ can give rise to level clustering which *for low-lying states* appears in the form of strong correlations between neighbouring levels and spacings (these structures correspond to classical closed orbits [3,4]). Such strong correlations can be seen in fig. 3 of ref. [1], calculated for an isotropic two-dimensional harmonic oscillator with quartic circularly-symmetric perturbing potential. This system separates in polar coordinates and gives rise to a function $E(m, n)$ whose contours are slightly concave towards the origin of the plane of quantum numbers $m, n$. As a model we can take [2]

$$E(m, n) = K\,[(m + \alpha n)^2 - \delta m^2]\ , \qquad (2)$$

where the normalization constant is

$$K = \frac{\arcsin([\delta/(1-\delta)]^{1/2})}{2\alpha\delta^{1/2}}, \qquad (3)$$

$\delta$ is the perturbation and $\alpha$ the frequency ratio which for an isotropic oscillator in polar action-angle variables equals $\frac{1}{2}$ (orbits are ellipses, so one rotation corresponds to two librations). Fig. 3 shows the spacings graph and correlation plot of the first 100 levels, for $\delta = 0.11012$. Evidently there is strong structure, resembling that seen in fig. 3 of ref. [1] and quite different from the random spacings of figs. 1a and 1b. But *for high-lying states* these local structures disappear as a result of the expansion and incoherent superposition of the clusters of levels as $n$ increases [2]; this is clear from fig. 4 which shows the spacings graph and correlation plot for levels 1700 to 1800, exhibiting the expected irregularity.

Fig. 4 of ref. [1] shows the spacings graph and correlation plot for a weakly nonintegrable system, in support of the claim that even very small zones of chaotic classical motion are revealed by irregularities in the quantum level spacings. The irregularities already presented (figs. 1a, 2a, 3, 4) for integrable systems demonstrate that there is no basis for this claim. Indeed, numerical evidence [5] and theoretical argument [6,7] indicate that the distribution of levels and spacings is *less* irregular for chaotic classical motion than for integrable classical motion. These remarks should not be misinterpreted as denying the existence of the "irregular spectrum" [8] corresponding to chaotic classical motion, but rather as a reminder that the irregularity is in the wave functions [9,10] and matrix elements [6], not the distribution of energy levels.

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