Fin Articles

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Green Eggs and Kelly

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The other night, I read Aristotle's *Lyceum 18: The Kelly Criterion* (http://www.wilmott.com/article.cfm?id=111) just before going to bed. Earlier I had put my daughter to sleep by reading Dr. Seuss' *Green Eggs and Ham*, and you can guess the result. All through the night, my dreams were haunted by someone named "Aristelli" who wanted me to bet like Kelly.

"Would you like to bet like Kelly?
Would you like that bet with jelly?
Would you like it on the telly?"

"No," I would reply, "I do not like the bet by Kelly.
Not on the telly or with jelly,
I do not like it Aristelli!"

This went on through "in a deli/with some corned beef in your belly," "with a score by Zeferelli/holding hands with Sue and Nelly," "eating spinach vermicelli" and anything else my subconscious could dredge up.

It's not that I'm against the Kelly criterion, it's an elegant solution, but to a problem no one has. I don't like it because it represents an important misconception about trading strategy, which is the essential distinction between a trader and a gambler (also, on the other side, between a trader and an investor).

Kelly says that if you want to maximize the expected growth rate of your wealth, when presented with a gamble, bet the fraction of your wealth equal to the expected log return of the bet divided by the variance of log return (despite the appearance, this is not a unit inconsistency, since log return is dimensionless, expected log return and variance of log return are both dimensionless and therefore the ratio is dimensionless).

The trouble is that there is no reason to want to maximize the expected growth rate of wealth, there is no reason to restrict betting fractions of wealth and the criterion cannot be extended to continuous time markets with many correlated bets available.

I'm going to start with the same problem, but not assume a goal (maximize expected growth rate of wealth), type of strategy (bet a constant fraction of wealth) or probability distribution. The important point I want to make is that you can get an answer without any of these things. Trading strategy does not depend on goals or probabilities, and you can't restrict yourself to a narrow range of tactics.

Let's say I am going to make a series of n bets. I choose the amounts of each bet, if I win I win that amount, if I lose, I pay that amount. The question is: How should I pick the amounts? The answer is: You're asking the wrong question.

There are 2^n possible outcomes of these bets. What you should be asking is: What set of outcome payoffs do I want? For example, if $n=2$, the possible outcomes are win-win, win-lose, lose-win and lose-lose. Win-lose and lose-win are different outcomes, they may have different probabilities (I don't assume that the probabilities of both bets are the same or that their outcomes are independent) and I may bet a different amount if I win the first bet rather than losing it. Pick four numbers for your payoffs in these four cases.

You can pick any set of payoffs for these 2^n outcomes, as long as they add up to zero. So you can choose the shape of your payoff distribution (or approximate it with 2^n dots, which should be sufficiently close if $n>10$, in real life n is very large since each tick of any financial asset is a potential bet). Once you pick the shape, the market will locate it for you (by insisting all the points add up to zero). But you can pick this shape without knowing anything about the probabilities of the bets nor the correlation among them.

Once you pick your payoffs, how do you set your individual bets? First you have to decide which payoff goes with which outcome. Here is where probabilities enter the problem. Obviously, you want the highest payoff to go to the outcome with highest probability, second highest payoff to outcome with second highest probability, and so on. So you have to rank the outcomes in order of probability. You do not need to know the absolute probabilities, just the rank order.

Once you match up the payoffs with the outcomes, take the average of all payoffs for outcomes that start with a win, and subtract the average of all payoffs for outcomes that start with a loss. Half that difference is your first bet. You need two second bets, one if you win the first, one if you lose it. To get the second bet after a win, take the average of all payoffs for outcomes that start with two wins, and subtract the average of all outcomes that start with a win followed by a loss. Half that difference is your second bet after a win. Continue in this manner and you have computed your entire betting strategy.

This does assume you have complete freedom in your bets, but restrictions are easily accommodated. For example, if you are forbidden to lose more than a maximum amount W (your total wealth, for example) then make the smallest payoff not less than $-W$. If there is a bet limit, make sure there is no gap in the ordered sequence of payoff amounts greater than twice the maximum bet. If you cannot short (that is, if the bets are offered one-sided) then you have to assign the payoffs to outcomes such that for any series of wins and losses ending in a win, the average of all payoffs for outcomes that begin with that series is not less than the average of all payoffs for outcomes starting with the same series changing the last result to loss.

For a specific example, let's make the usual Kelly assumptions. Suppose we want to maximize expected value of log wealth (this corresponds to log utility which is a notoriously bad utility function model). Let $m=2^n$. I want to maximize the sum from 0 to m-1 of $p_i\ln(W+X_i)$, where p_i is the probability of outcome i, W is initial wealth and X_i is the payoff, the winnings from the bets.

Since the sum of the X_i must be 0, we rewrite the sum above from 1 to m and add $p_0\ln(W-X_1-X_2-...-X_{m-1})$. The derivative of this function with respect to X_i is $-p_0/(W+X_0)+p_i/(W+X_i)$ which equals zero if $X_i=p_i(W+X_0)/p_0-W$. The sum of these for all i is $(W+X_0)/p_0-mW$ which equals zero if $X_0=(mp_0-1)W$. That allows us to rewrite the formula for X_i as $(mp_i-1)W$.

Notice that our utility function alone has determined our set of outcomes, but we need probability assumptions to convert this to a betting strategy. Let's follow Kelly some more and assume that each bet is independent, with constant probability p of winning. All outcomes with the same number of wins are equally likely. For k wins the probability is $(2p)^k(2-2p)^{n-k}$ so we assign the outcome the payoff $[((2p)^k(2-2p)^{n-k}-1]W$.

What's our first bet? The average payoff for outcomes that start with a win is $(2p-1)W$ (this is easy to see because the sum of their p_i is p) while those that start with a loss average $(1-2p)W$. Half the difference is $(2p-1)W$ and that is our first bet. The symmetry of the solution, and since it does not involve m, makes it clear that the solution is always to bet $(2p-1)$ times current wealth. Kelly recommends the mean, $2p-1$, times W divided by the variance, $4p(1-p)$. The last term is near 1 if p is near 0.5. Our solution is exact, Kelly is an approximation that only works if p is near 0.5 (more generally, if the mean is small relative to the standard deviation).

Is that why I prefer my derivation, exact versus approximate? No. Is it that my version can incorporate any utility function, not just log, and any set of outcome probabilities? No again. Is it that it justifies the strategy of betting a constant fraction of wealth instead of assuming it? Still no, although I would make the point that it is easy to solve this problem for the much more reasonable negative exponential utility, $U(w)=\exp(-aw)$. Our derivative with respect to X_i becomes $ap_0\exp(-aX_0)-ap_i\exp(-aX_i)$ which equals zero if $X_i= X_0-[\ln(p_0)-\ln(p_i)]/a$. Adding these up and setting the total to zero simplifies the expression to $X_i= [\ln(p_i)-q]/a$ where q is the average of all the $\ln(p_i)$'s. For the i.i.d. Bernoulli example, bet $[\ln(p)-\ln(1-p)]/2a$ each time. If we set $a=1/W$ and p is near 0.5, this is similar to the result for log utility, except that the bet size does not change with wealth.

I like my derivation because it decomposes the problem the way any trading organization should. You start by deciding what set of payoffs you want. You, not the market. You don't ask what profit opportunities there will be, what the probabilities of success are or how big the risk is. You don't know much about any of these things. But that's no reason to give up on calculation. You pick the shape of your outcome distribution and let the market locate it for you (by making everything add up to your expected return). This separation of the problem into outcome distribution shape and probability assignment is critical for understanding financial markets and running an efficient trading organization.

This is why banks have risk managers. I can set the outcome distribution for the bank, without worry about the specifics bets available in each market. I certainly don't have to estimate the probabilities of various scenarios, nor the expected return and risk of any trader's strategy. I take the distribution of outcomes the bank wants, and break them down into tools so that each trader knows exactly how much to bet on each opportunity.

Once you know the payoffs you want, you have to assign them to specific outcomes. Now probabilities enter the problem, but only in relative terms. This decision is made by the trader. But she doesn't have to estimate a probability, she just has to take her limit and allocate it to the best opportunities she can find. Her boss has to decide how to take a bigger limit and allocate it to the better traders. His boss does the same, up to the head of the entire trading organization. Traders make decisions about relative probabilities, risk management chooses how to turn those decisions into P&L for the bank as a whole.

The Kelly Criterion started out by saying gamblers and traders are similar. This is only true up to a point. It's like saying soldiers, hunters, police officers and bank robbers are all similar because they use guns. It's true that all of these people need some familiarity with firearms. But they use different types in different ways for different purposes. A bank robber hopes never to fire at all, and is little concerned about what he hits if he does fire. Hunters and police officers practice a lot because on the rare occasions they shoot for real (the average hunter makes a kill shot less than once every two years, the average police office never fires throughout his career), it is extremely important to hit what they aim at and not what they don't aim at. Soldiers fire a lot, and care little about what they hit, most of the time they are simply trying to force an enemy under cover for some larger purpose (in fact, a famous study found that the majority of soldiers deliberately try to miss the enemy, because if you try to hit them, they are forced to try to hit you; if both sides fire up in the air, the battle's result is the same but fewer people get hurt).

Gamblers and traders both make bets but their constraints are very different. There is no time element to a gambler's strategy, while time is essential in finance. If you bet on the stock market, you have an excess expected log return of about 4% per year, with a standard deviation of 20%. But you have to wait a year to find out the payoff, you can't play 100 hands of stock market in an evening, for an expected excess log return of 400% with a standard deviation of 200%. You need 100 years to place that bet.

Therefore, traders care about risk. They can't wait 100 years for it to diversify away, they need profits quarterly to keep their jobs/bonuses/investors. The reason a trader does not bet the limit on every positive expected value opportunity that comes up is that strategy leads to a quarterly distribution of returns that is too volatile to attract money. Moreover, no one can tell if you're lucky or good (not even yourself) if you cannot control your distribution of returns.

Gamblers care about risk for a different reason. They can in fact diversify risk away just by playing longer. However, they care not only about their terminal value of wealth, but about fluctuations along the way. In fact, a pure gambler cares nothing for terminal value of wealth because he will never stop gambling. It's the ups and downs that are the entire point of the activity. Also, you never want to be down so much you have to stop gambling because then all your future positive expected value, plus all your excitement, go away.

At the other extreme is an investor. He cares about terminal wealth with a much longer horizon than a trader, because he's playing with his own money. Suppose you found a stock that sells for $20 today, will be $100 in ten years, but might fall to $5 in between. An investor doesn't care about the risk of the $5 intermediate price, only the final price matters. A gambler cares just to make sure a fall to $5 won't wipe him out. A trader thinks you have suggested two strategies, short from $20 to $5, then long from $5 to $100.

Another reason an investor is different is he has higher transaction costs. In our formulation of the problem, we assumed any bet could be made at any time, there was no cost to changing the level of bet. This is reasonable for gamblers and traders. But investors save money and trouble by buy-and-hold, which amounts to making the same bet each time. Therefore investors often take the return distribution the market gives them, or modify it only by selecting different securities or simple management strategies. Given their longer time horizon, greater capital and less aggressive strategies than traders; it is reasonable for investors to assume the shape of the terminal return distribution doesn't matter very much.

Problems in financial market often result from traders acting like gamblers or investors. When a trader is near to losing his job, or his hedge fund, he has to worry about bet-by-bet risk, which makes him act like a gambler. On the other hand when traders get too much of their own capital, or their reputation insulates them from penalties for losses, they can start behaving like investors.

I do not like green eggs and Kelly,
I will not put them in my belly,
So do not bug me Aristelli.

posted by Gamma @ 1:36 AM 0 comments

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