4th order GL corrections due to interaction of particles with density modulations.

$\varepsilon_k = k^2/2 - 1/2$ $(k_F = 1)$; $n_F(\varepsilon_k) = 1/[1 + e^{\varepsilon_k/T}]$

$$\begin{aligned}
\Delta\Omega_B^{(4)} &= \tfrac{v^4\|\rho_q\|^4}{4\beta}\textstyle\sum_{\omega_n k}\tfrac{1}{(i\omega_n-\varepsilon_k)^2}\tfrac{1}{(i\omega_n-\varepsilon_{k+q})^2}\\
&= \tfrac{v^4\|\rho_q\|^4}{4}\textstyle\sum_k\int dz\tfrac{n_F(z)}{(z-\varepsilon_k)^2(z-\varepsilon_{k+q})^2}\\
&= \tfrac{v^4\|\rho_q\|^4}{4}\textstyle\sum_k\left[\tfrac{n_F(z)}{(z-\varepsilon_k)^2}\right]'_{\varepsilon_{k+q}}+\left[\tfrac{n_F(z)}{(z-\varepsilon_{k+q})^2}\right]'_{\varepsilon_k}\\
&= \tfrac{v^4\|\rho_q\|^4}{4}\textstyle\sum_k\tfrac{n_F'(\varepsilon_k)+n_F'(\varepsilon_{k+q})}{(\varepsilon_{k+q}-\varepsilon_k)^2}-2\tfrac{n_F(\varepsilon_k)-n_F(\varepsilon_{k+q})}{(\varepsilon_k-\varepsilon_{k+q})^3}\\
&= \tfrac{v^4\|\rho_q\|^4}{2}\textstyle\sum_k -2\tfrac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^3}+\tfrac{n_F'(\varepsilon_k)}{(\varepsilon_{k+q}-\varepsilon_k)^2}\\
&\equiv \tfrac{v^4\|\rho_q\|^4}{2}(B_1+B_2)
\end{aligned}$$

$$B_1 = \textstyle\sum_k -2\tfrac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^3}$$
$$B_2 = \textstyle\sum_k \tfrac{n_F'(\varepsilon_k)}{(\varepsilon_{k+q}-\varepsilon_k)^2}$$

$$\begin{aligned}
B_1 &= \textstyle\sum_k -2\tfrac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^3}\\
&= \tfrac{2}{(2\pi)^2}\textstyle\int kdkd\theta\tfrac{n_F(\varepsilon_k)}{(kq\cos\theta+q^2/2)^3}\\
&= \tfrac{16}{(2\pi)^2}\textstyle\int kdkn_F(\varepsilon_k)X(k,q)
\end{aligned}$$

$$\begin{aligned}
X(k,q) &= \int d\theta\frac{1}{(2kq\cos\theta+q^2)^3}\\
&\underset{\{z=e^{i\theta}\}}{=}\int(dz/iz)\frac{1}{(kq(z+1/z)+q^2)^3}\\
&= \int\frac{(dz/i)}{(kq)^3}\frac{z^2}{(z^2+1+zq/k)^3}\\
&= -\frac{i}{(kq)^3}\int dz\frac{z^2}{(z-z_1)^3(z-z_2)^3}\\
&\quad\text{where}\left\{z_{1,2} = -\frac{q}{2k}\pm\sqrt{(q/2k)^2-1}\right\}\qquad z_1z_2 = 1\\
&\quad |z_1|<1\text{ and }|z_2|>1\text{ for }q>2k\\
&= \frac{2\pi}{(kq)^3}\frac{1}{2}\left[\frac{z^2}{(z-z_2)^3}\right]''_{z_1}\\
&= \frac{\pi}{(kq)^3}\left[\frac{12z_1^2}{(z_1-z_2)^5}-\frac{12z_1}{(z_1-z_2)^4}+\frac{2}{(z_1-z_2)^3}\right]\\
&= \frac{\pi}{(kq)^3}\left[\frac{12z_1z_2}{(z_1-z_2)^5}+\frac{2}{(z_1-z_2)^3}\right]
\end{aligned}$$

$$
\begin{aligned}
B_1 = \quad & \frac{16}{(2\pi)^2}\int kdkn_F(\varepsilon_k)\frac{2\pi}{(kq)^3}\left[\frac{6}{(z_1-z_2)^5}+\frac{1}{(z_1-z_2)^3}\right] \\
= \quad & \frac{16}{(2\pi)}\int kdkn_F(\varepsilon_k)\frac{1}{(kq)^3}\left[\frac{6}{32[(q/2k)^2-1]^{5/2}}+\frac{1}{8[(q/2k)^2-1]^{3/2}}\right] \\
& \frac{1}{(2\pi)}\int kdkn_F(\varepsilon_k)\frac{1}{(q)^3}\left[\frac{3(k^2\pm(q/2)^2)}{[(q/2)^2-k^2]^{5/2}}+\frac{2}{[(q/2)^2-k^2]^{3/2}}\right] \\
& \frac{1}{(2\pi)}\int kdkn_F(\varepsilon_k)\frac{1}{(q)^3}\left[\frac{3(q/2)^2}{[(q/2)^2-k^2]^{5/2}}-\frac{1}{[(q/2)^2-k^2]^{3/2}}\right] \\
=_{\circ T=0} \quad & \frac{1}{4\pi q^3}\int dx\Theta(0<x<k_F^2)\left[\frac{3(q/2)^2}{[(q/2)^2-x]^{5/2}}-\frac{1}{[(q/2)^2-x]^{3/2}}\right] \\
= \quad & \frac{1}{4\pi q^3}\left[\frac{2}{3}\frac{3(q/2)^2}{[(q/2)^2-x]^{3/2}}-2\frac{1}{[(q/2)^2-x]^{1/2}}\right]_0^{k_F^2} \\
= \quad & \frac{1}{4\pi q^3}\left[2\left\{\frac{(q/2)^2}{[(q/2)^2-k_F^2]^{3/2}}-\frac{1}{(q/2)}\right\}-2\left\{\frac{1}{[(q/2)^2-k_F^2]^{1/2}}-\frac{1}{(q/2)}\right\}\right] \\
= \quad & \frac{1}{2\pi q^3}\left[\frac{(q/2)^2}{[(q/2)^2-k_F^2]^{3/2}}-\frac{1}{[q^2-4k_F^2]^{1/2}}\right] \\
= \quad & \frac{4}{\pi q^3}\frac{k_F^2}{[q^2-4k_F^2]^{3/2}}
\end{aligned}
$$

$$
\begin{aligned}
B_2 = \quad & \sum_k \frac{n_F'(\varepsilon_k)}{(\varepsilon_{k+q}-\varepsilon_k)^2} \\
= \quad & \frac{1}{(2\pi)^2}\int kdkd\theta\frac{n_F'(\varepsilon_k)}{(kq\cos\theta+q^2/2)^2} \\
= \quad & \frac{1}{\pi^2}\int kdk(dz/i)\frac{1}{(kq)^2}\frac{zn_F'(\varepsilon_k)}{(z^2+1+(q/k)z)^2} \\
= \quad & \frac{1}{\pi^2}\int kdk\frac{n_F'(\varepsilon_k)}{(kq)^2}\frac{(dz/i)z}{(z-z_1)^2(z-z_2)^2} \\
= \quad & \frac{1}{\pi^2}\int kdk\frac{n_F'(\varepsilon_k)}{(kq)^2}(2\pi i/i)\left[\frac{z}{(z-z_2)^2}\right]'_{z_1} \qquad |z_1|<1 \text{ and } |z_2|>1 \text{ for } q>2k \\
& \to \left[\frac{z}{(z-z_2)^2}\right]'_{z_1} = \frac{1}{(z_1-z_2)^2}-\frac{2z_1}{(z_1-z_2)^3} = -\frac{z_1+z_2}{(z_1-z_2)^3} \\
& \qquad = \frac{q/k}{8[(q/2k)^2-1]^{3/2}} \\
= \quad & \frac{2}{\pi}\int kdk\frac{n_F'(\varepsilon_k)}{(kq)^2}\frac{q/k}{8[(q/2k)^2-1]^{3/2}} \\
= \quad & \frac{1}{\pi q}\int d(k^2)\frac{n_F'(\varepsilon_k)}{1}\frac{1}{8[(q/2)^2-k^2]^{3/2}} \\
= \quad & -\frac{2}{\pi q}\frac{1}{[q^2-4k_F^2]^{3/2}}
\end{aligned}
$$

Now, putting the B1 and B2 terms together,

$$
\begin{aligned}
\Delta\Omega_B^{(4)} = \quad & \frac{v^4\|\rho_q\|^4}{2}(B_1+B_2) \\
= \quad & \frac{v^4\|\rho_q\|^4}{2}\left(\frac{4}{\pi q^3}\frac{k_F^2}{[q^2-4k_F^2]^{3/2}}-\frac{2}{\pi q}\frac{1}{[q^2-4k_F^2]^{3/2}}\right) \\
= \quad & \frac{v^4\|\rho_q\|^4}{4(2\pi)^2}\times\frac{16\pi}{q^3}\frac{2k_F^2-q^2}{[q^2-4k_F^2]^{3/2}}
\end{aligned}
$$

The expression after x sign corresponds very well to the one obtained by means of direct numerical integration with regularization, in attached matlab code. This expression here is derived for $q > 2k_F$. Note that in this regime it is always attractive. For $q < 2k_F$the countour integration is tricky, since both poles during angular integration are located on unit circle. This may indicate that the interaction is divergent for any $q < 2k_F$ as $T \to 0$. Numerics at finite temperature indicate that after going through singularity at $q = 2k_F$ this interaction term switches to repulsion at $q < 2k_F$.