4th order GL corrections due to interaction of particles with density modulations.

$\varepsilon_k = k^2/2 - 1/2$ ($k_F = 1$); $n_F(\varepsilon_k) = 1/[1 + e^{\varepsilon_k/T}]$

********* "A" terms in 4th order energy *******************

$k \to k+q \to k+2q \to k+q \to k$

$E_1 \to E_2 \;\to\; E_3 \;\to\; E_2 \to E_1$

$$
\begin{aligned}
\Delta\Omega_A^{(4)} &= \frac{v^4\|\rho_q\|^4}{4\beta}\sum_{\omega_n k}\frac{1}{(i\omega_n-\varepsilon_k)}\frac{1}{(i\omega_n-\varepsilon_{k+q})^2}\frac{1}{(i\omega_n-\varepsilon_{k+2q})} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\int_C dz\frac{1}{(z-\varepsilon_k)}\frac{n_F(z)}{(z-\varepsilon_{k+q})^2}\frac{1}{(z-\varepsilon_{k+2q})} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^2(\varepsilon_k-\varepsilon_{k+2q})} \\
&\quad+\frac{n_F(\varepsilon_{k+2q})}{(\varepsilon_{k+2q}-\varepsilon_k)(\varepsilon_{k+2q}-\varepsilon_{k+q})^2} \\
&\quad+\frac{n'_F(\varepsilon_{k+q})}{(\varepsilon_{k+q}-\varepsilon_k)(\varepsilon_{k+q}-\varepsilon_{k+2q})} \qquad \leftarrow \left\{\frac{n_F(z)}{(z-\varepsilon_k)(z-\varepsilon_{k+2q})}\right\}'_{\varepsilon_{k+q}} \\
&\quad-\frac{n_F(\varepsilon_{k+q})}{(\varepsilon_{k+q}-\varepsilon_k)^2(\varepsilon_{k+q}-\varepsilon_{k+2q})} \\
&\quad-\frac{n_F(\varepsilon_{k+q})}{(\varepsilon_{k+q}-\varepsilon_k)(\varepsilon_{k+q}-\varepsilon_{k+2q})^2} \\
&\qquad\cdots\cdots\cdots\cdots \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{-n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^2(\varepsilon_k-\varepsilon_{k+2q})} \qquad \leftarrow \text{shift of } k \\
&\quad+\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k-2q})(\varepsilon_k-\varepsilon_{k-q})^2} \\
&\quad+\frac{n'_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k-q})(\varepsilon_k-\varepsilon_{k+q})} \\
&\quad-\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k-q})^2(\varepsilon_k-\varepsilon_{k+q})} \\
&\quad-\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k-q})(\varepsilon_k-\varepsilon_{k+q})^2} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^2(\varepsilon_k-\varepsilon_{k+2q})} \qquad \leftarrow (k\to -k) \\
&\quad+\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+2q})(\varepsilon_k-\varepsilon_{k+q})^2} \\
&\quad+\frac{n'_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})(\varepsilon_k-\varepsilon_{k-q})} \\
&\quad-\frac{2n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^2(\varepsilon_k-\varepsilon_{k-q})} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{2n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^2(\varepsilon_k-\varepsilon_{k+2q})} \\
&\quad-\frac{2n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^2(\varepsilon_k-\varepsilon_{k-q})} \\
&\quad+\frac{n'_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})(\varepsilon_k-\varepsilon_{k-q})}
\end{aligned}
$$

Attempt at alanytics

$$
\begin{aligned}
\Delta\Omega_A^{(4)} = & \frac{v^4\|\rho_q\|^4}{4}\sum_k \frac{-n_F(\varepsilon_k)}{(kq\cos\theta+q^2/2)^2(kq\cos\theta+q^2)} && \leftarrow (\varepsilon_k-\varepsilon_{k+q}) = -(kq\cos\theta+q^2/2)\\
& -\frac{n_F'(\varepsilon_k)}{(kq\cos\theta+q^2/2)(kq\cos\theta-q^2/2)} && \leftarrow (\varepsilon_k-\varepsilon_{k+2q}) = -2(kq\cos\theta+q^2)\\
& -\frac{2n_F(\varepsilon_k)}{(kq\cos\theta+q^2/2)^2(kq\cos\theta-q^2/2)}\\
= & \frac{v^4\|\rho_q\|^4}{4}\sum_k -\frac{8n_F(\varepsilon_k)}{(2kq\cos\theta+q^2)^2(2kq\cos\theta+2q^2)}\\
& -\frac{4n_F'(\varepsilon_k)}{(2kq\cos\theta+q^2)(2kq\cos\theta-q^2)}\\
& -\frac{16n_F(\varepsilon_k)}{(2kq\cos\theta+q^2)^2(2kq\cos\theta-q^2)}\\
= & -\frac{v^4\|\rho_q\|^4}{4(2\pi)^2}\int kdk\frac{1}{(kq)^3}\int(dz/i)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
& 8n_F(\varepsilon_k)\frac{z^2}{(z^2+1+zq/k)^2(z^2+1+2zq/k)} && \leftarrow A_1\\
& +16n_F(\varepsilon_k)\frac{z^2}{(z^2+1+zq/k)^2(z^2+1-zq/k)} && \leftarrow A_2\\
& +4kqn_F'(\varepsilon_k)\frac{z}{(z^2+1+zq/k)(z^2+1-zq/k)} && \leftarrow A_3
\end{aligned}
$$

********* "B" terms in 4th order energy *******************

$k \to k+q \to k \to k+q \to k$

$E_1 \to E_2 \to E_1 \to E_2 \to E_1$

$$
\begin{aligned}
\Delta\Omega_B^{(4)} = & \frac{v^4\|\rho_q\|^4}{4\beta}\sum_{\omega_n k}\frac{1}{(i\omega_n-\varepsilon_k)^2}\frac{1}{(i\omega_n-\varepsilon_{k+q})^2}\\
= & \frac{v^4\|\rho_q\|^4}{4}\sum_k\int_C dz\frac{n_F(z)}{(z-\varepsilon_k)^2(z-\varepsilon_{k+q})^2}\\
= & \frac{v^4\|\rho_q\|^4}{4}\sum_k\left[\frac{n_F(z)}{(z-\varepsilon_k)^2}\right]'_{\varepsilon_{k+q}}+\left[\frac{n_F(z)}{(z-\varepsilon_{k+q})^2}\right]'_{\varepsilon_k}\\
= & \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{n_F'(\varepsilon_k)+n_F'(\varepsilon_{k+q})}{(\varepsilon_{k+q}-\varepsilon_k)^2}-2\frac{n_F(\varepsilon_k)-n_F(\varepsilon_{k+q})}{(\varepsilon_k-\varepsilon_{k+q})^3}\\
= & \frac{v^4\|\rho_q\|^4}{2}\sum_k -2\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^3}+\frac{n_F'(\varepsilon_k)}{(\varepsilon_{k+q}-\varepsilon_k)^2}\\
\equiv & \frac{v^4\|\rho_q\|^4}{2}(B_1+B_2)
\end{aligned}
$$

$$
B_1 = \sum_k -2\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^3}
$$

$$
B_2 = \sum_k \frac{n_F'(\varepsilon_k)}{(\varepsilon_{k+q}-\varepsilon_k)^2}
$$

$$B_1 = \sum_k -2\frac{n_F(\varepsilon_k)}{(\varepsilon_k-\varepsilon_{k+q})^3}$$

$$= \frac{2}{(2\pi)^2}\int kdkd\theta \frac{n_F(\varepsilon_k)}{(kq\cos\theta+q^2/2)^3}$$

$$= \frac{16}{(2\pi)^2}\int kdkn_F(\varepsilon_k)X(k,q)$$

$$X(k,q) = \int d\theta \frac{1}{(2kq\cos\theta+q^2)^3}$$

$$\underset{\{z=e^{i\theta}\}}{=} \int (dz/iz)\frac{1}{(kq(z+1/z)+q^2)^3}$$

$$= \int \frac{(dz/i)}{(kq)^3}\frac{z^2}{(z^2+1+zq/k)^3}$$

$$= -\frac{i}{(kq)^3}\int dz \frac{z^2}{(z-z_1)^3(z-z_2)^3}$$

$$\text{where}\left\{z_{1,2} = -\frac{q}{2k} \pm \sqrt{(q/2k)^2-1}\right\} \qquad z_1z_2 = 1$$

$|z_1| < 1$ and $|z_2| > 1$ for $q > 2k$

$$= \frac{2\pi}{(kq)^3}\frac{1}{2}\left[\frac{z^2}{(z-z_2)^3}\right]''_{z_1}$$

$$= \frac{\pi}{(kq)^3}\left[\frac{12z_1^2}{(z_1-z_2)^5} - \frac{12z_1}{(z_1-z_2)^4} + \frac{2}{(z_1-z_2)^3}\right]$$

$$= \frac{\pi}{(kq)^3}\left[\frac{12z_1z_2}{(z_1-z_2)^5} + \frac{2}{(z_1-z_2)^3}\right]$$

$$B_1 = \frac{16}{(2\pi)^2}\int kdkn_F(\varepsilon_k)\frac{2\pi}{(kq)^3}\left[\frac{6}{(z_1-z_2)^5} + \frac{1}{(z_1-z_2)^3}\right]$$

$$= \frac{16}{(2\pi)}\int kdkn_F(\varepsilon_k)\frac{1}{(kq)^3}\left[\frac{6}{32[(q/2k)^2-1]^{5/2}} + \frac{1}{8[(q/2k)^2-1]^{3/2}}\right]$$

$$\frac{1}{(2\pi)}\int kdkn_F(\varepsilon_k)\frac{1}{(q)^3}\left[\frac{3(k^2\pm(q/2)^2)}{[(q/2)^2-k^2]^{5/2}} + \frac{2}{[(q/2)^2-k^2]^{3/2}}\right]$$

$$\frac{1}{(2\pi)}\int kdkn_F(\varepsilon_k)\frac{1}{(q)^3}\left[\frac{3(q/2)^2}{[(q/2)^2-k^2]^{5/2}} - \frac{1}{[(q/2)^2-k^2]^{3/2}}\right]$$

$$=_{\circ T=0} \frac{1}{4\pi q^3}\int dx\Theta(0<x<k_F^2)\left[\frac{3(q/2)^2}{[(q/2)^2-x]^{5/2}} - \frac{1}{[(q/2)^2-x]^{3/2}}\right]$$

$$= \frac{1}{4\pi q^3}\left[\frac{2}{3}\frac{3(q/2)^2}{[(q/2)^2-x]^{3/2}} - 2\frac{1}{[(q/2)^2-x]^{1/2}}\right]_0^{k_F^2}$$

$$= \frac{1}{4\pi q^3}\left[2\left\{\frac{(q/2)^2}{[(q/2)^2-k_F^2]^{3/2}} - \frac{1}{(q/2)}\right\} - 2\left\{\frac{1}{[(q/2)^2-k_F^2]^{1/2}} - \frac{1}{(q/2)}\right\}\right]$$

$$= \frac{1}{2\pi q^3}\left[\frac{(q/2)^2}{[(q/2)^2-k_F^2]^{3/2}} - \frac{1}{[q^2-4k_F^2]^{1/2}}\right]$$

$$= \frac{4}{\pi q^3}\frac{k_F^2}{[q^2-4k_F^2]^{3/2}}$$

$$
\begin{aligned}
B_2 = &\quad \sum_k \frac{n'_F(\varepsilon_k)}{(\varepsilon_{k+q}-\varepsilon_k)^2} \\
= &\quad \frac{1}{(2\pi)^2}\int kdkd\theta \frac{n'_F(\varepsilon_k)}{(kq\cos\theta + q^2/2)^2} \\
= &\quad \frac{1}{\pi^2}\int kdk(dz/i)\frac{1}{(kq)^2}\frac{zn'_F(\varepsilon_k)}{(z^2+1+(q/k)z)^2} \\
= &\quad \frac{1}{\pi^2}\int kdk\frac{n'_F(\varepsilon_k)}{(kq)^2}\frac{(dz/i)z}{(z-z_1)^2(z-z_2)^2} \\
= &\quad \frac{1}{\pi^2}\int kdk\frac{n'_F(\varepsilon_k)}{(kq)^2}(2\pi i/i)\left[\frac{z}{(z-z_2)^2}\right]'_{z_1} \qquad |z_1|<1 \text{ and } |z_2|>1 \text{ for } q>2k \\
&\to \left[\frac{z}{(z-z_2)^2}\right]'_{z_1} = \quad \frac{1}{(z_1-z_2)^2} - \frac{2z_1}{(z_1-z_2)^3} = -\frac{z_1+z_2}{(z_1-z_2)^3} \\
&\qquad = \quad \frac{q/k}{8[(q/2k)^2-1]^{3/2}} \\
= &\quad \frac{2}{\pi}\int kdk\frac{n'_F(\varepsilon_k)}{(kq)^2}\frac{q/k}{8[(q/2k)^2-1]^{3/2}} \\
= &\quad \frac{1}{\pi q}\int d(k^2)\frac{n'_F(\varepsilon_k)}{1}\frac{1}{8[(q/2)^2-k^2]^{3/2}} \\
= &\quad -\frac{2}{\pi q}\frac{1}{[q^2-4k_F^2]^{3/2}}
\end{aligned}
$$

Now, putting the B1 and B2 terms together,

$$
\begin{aligned}
\Delta\Omega_B^{(4)} = &\quad \frac{v^4\|\rho_q\|^4}{2}(B_1+B_2) \\
= &\quad \frac{v^4\|\rho_q\|^4}{2}\left(\frac{4}{\pi q^3}\frac{k_F^2}{[q^2-4k_F^2]^{3/2}} - \frac{2}{\pi q}\frac{1}{[q^2-4k_F^2]^{3/2}}\right) \\
= &\quad \frac{v^4\|\rho_q\|^4}{4(2\pi)^2} \times \frac{16\pi}{q^3}\frac{2k_F^2-q^2}{[q^2-4k_F^2]^{3/2}}
\end{aligned}
$$

The expression after $\times$ corresponds very well to the one obtained by means of direct numerical integration. This expression here is derived for $q > 2k_F$. Note that in this regime it is always attractive. For $q < 2k_F$the countour integration is tricky, since both poles during angular integration are located on unit circle. Numerics at finite temperature indicate that after going through singularity at $q = 2k_F$ this interaction term switches to repulsion at $q < 2k_F$, remaining finite as $T \to 0$.

General General 4th order terms, and their relationship to self-interaction

"Diamond" term

$k \to k + Q_1 \to k + Q_1 + Q_2 \to k + Q_2 \to k$

$E_1 \to E_2 \to \qquad E_3 \to \qquad\qquad E_4 \to \quad E_1$

$$
\begin{aligned}
\Delta\Omega_D^{(4)} &= \frac{v^4\|\rho_q\|^4}{4\beta}\sum_{\omega_n k}\frac{1}{(i\omega_n-E_1)}\frac{1}{(i\omega_n-E_2)(i\omega_n-E_3)}\frac{1}{(i\omega_n-E_4)} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\int_C dz\,\frac{n_F(z)}{(z-E_1)(z-E_2)(z-E_3)(z-E_4)} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\Big\{\frac{n_F(E_1)}{(E_1-E_2)(E_1-E_3)(E_1-E_4)} \\
&\quad+\frac{n_F(E_2)}{(E_2-E_1)(E_2-E_3)(E_2-E_4)} \\
&\quad+\frac{n_F(E_3)}{(E_3-E_1)(E_3-E_2)(E_3-E_4)} \\
&\quad+\frac{n_F(E_4)}{(E_4-E_1)(E_4-E_2)(E_4-E_3)}\Big\}
\end{aligned}
$$

"V" term

$k \to k+Q_1 \to k \to k+Q_2 \to k$

$E_1 \to E_2 \to \ldots . E_1 \to E_4 \to \ldots E_1$

$$
\begin{aligned}
\Delta\Omega_V^{(4)} &= \frac{v^4\|\rho_q\|^4}{4\beta}\sum_{\omega_n k}\frac{1}{(i\omega_n-E_1)^2(i\omega_n-E_2)(i\omega_n-E_4)} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\int_C dz\,\frac{n_F(z)}{(z-E_1)^2(z-E_2)(z-E_4)} \\
&= \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{n_F(E_2)}{(E_1-E_2)^2(E_2-E_4)} \\
&\quad-\frac{n_F(E_4)}{(E_1-E_4)^2(E_2-E_4)} \\
&\quad+\frac{n'_F(E_1)}{(E_1-E_2)(E_1-E_4)} \\
&\quad-\frac{n_F(E_1)}{(E_1-E_2)^2(E_1-E_4)} \\
&\quad-\frac{n_F(E_1)}{(E_1-E_2)(E_1-E_4)^2}
\end{aligned}
$$

0.1 Connection

These can be related to the A and B terms in the self-interaction, $D \to A$ and $V \to B$,in the limit $Q_1 \to Q_2$, i.e. $E_2 \to E_4$. Indeed,

$$
\begin{aligned}
\Delta\Omega_A^{(4)} &= \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{n_F(E_1)}{(E_1-E_2)^2(E_1-E_3)} \\
&\quad+\frac{n_F(E_3)}{(E_3-E_1)(E_3-E_2)^2} \\
&\quad+\frac{n'_F(E_2)}{(E_2-E_1)(E_2-E_3)} \\
&\quad-\frac{n_F(E_2)}{(E_2-E_1)^2(E_2-E_3)} \\
&\quad-\frac{n_F(E_2)}{(E_2-E_1)(E_2-E_3)^2}
\end{aligned}
$$

$$
\Delta\Omega_B^{(4)} = \frac{v^4\|\rho_q\|^4}{4}\sum_k\frac{n'_F(E_1)+n'_F(E_2)}{(E_2-E_1)^2} - 2\frac{n_F(E_1)-n_F(E_2)}{(E_1-E_2)^3}
$$

While, as $E_2 - E_4 \to 0$:

$$\begin{aligned}
\Delta\Omega_D^{(4)} = \quad & \frac{v^4\|\rho_q\|^4}{4}\sum_k \frac{n_F(E_1)}{(E_1-E_2)(E_1-E_3)(E_1-E_4)} \\
& + \frac{n_F(E_3)}{(E_3-E_1)(E_3-E_2)(E_3-E_4)} \\
& + \frac{1}{E_4-E_2}\left[\frac{n_F(E_4)}{(E_4-E_1)(E_4-E_3)} - \frac{n_F(E_2)}{(E_2-E_1)(E_2-E_3)}\right] \\
\Rightarrow \quad & \frac{v^4\|\rho_q\|^4}{4}\sum_k \frac{n_F(E_1)}{(E_1-E_2)^2(E_1-E_3)} \\
& + \frac{n_F(E_3)}{(E_3-E_1)(E_3-E_2)^2} \\
& + \left[\frac{n_F(w)}{(w-E_1)(w-E_3)}\right]'_{E_2} \qquad \Rightarrow \Delta\Omega_A^{(4)}
\end{aligned}$$

$$\begin{aligned}
\Delta\Omega_V^{(4)} = \quad & \frac{v^4\|\rho_q\|^4}{4}\sum_k \frac{1}{E_4-E_2}\left[\frac{n_F(E_4)}{(E_4-E_1)^2} - \frac{n_F(E_2)}{(E_2-E_1)^2}\right] \\
& + \frac{n'_F(E_1)}{(E_1-E_2)(E_1-E_4)} \\
& - \frac{n_F(E_1)}{(E_1-E_2)^2(E_1-E_4)} \\
& - \frac{n_F(E_1)}{(E_1-E_2)(E_1-E_4)^2} \\
\Rightarrow \quad & \frac{v^4\|\rho_q\|^4}{4}\sum_k \left[\frac{n_F(w)}{(w-E_1)^2}\right]'_{E_2} \\
& + \frac{n'_F(E_1)}{(E_1-E_2)^2} \\
& - \frac{2n_F(E_1)}{(E_1-E_2)^3} \qquad \Rightarrow \Delta\Omega_B^{(4)}
\end{aligned}$$

So, indeed we find that diagrams for self and mutual interactions are smoothly connected at finite temperatures.

****** Absence of divergences at non-zero T. ***************

For $k_F < q < 2k_F$, Ω_A term is singular if e.g. $E_1 \approx E_2$ (note that E_1in this case cannot be close to E_3). Expand carefully around this point:

$$\begin{aligned}
\Delta\Omega_A^{(4)} \sim & \sum_k \tfrac{n_F(E_1)}{(E_1-E_2)^2(E_1-E_3)} \\
& + \tfrac{n_F(E_3)}{(E_3-E_1)(E_3-E_2)^2} \\
& + \tfrac{n'_F(E_2)}{(E_2-E_1)(E_2-E_3)} \\
& - \tfrac{n_F(E_2)}{(E_2-E_1)^2(E_2-E_3)} \\
& - \tfrac{n_F(E_2)}{(E_2-E_1)(E_2-E_3)^2} \\
= & \sum_k \tfrac{n_F(E_1)}{(E_1-E_2)^2} \left(\tfrac{1}{(E_1-E_3)} \pm \tfrac{1}{(E_2-E_3)}\right) \qquad \to \frac{E_2 - E_1}{(E_1-E_3)(E_2-E_3)} + \frac{1}{(E_2-E_3)} \\
& - \tfrac{n_F(E_2)+n'_F(E_2)(E_1-E_2)}{(E_1-E_2)^2(E_2-E_3)} \\
& - \tfrac{n_F(E_2)}{(E_2-E_1)(E_2-E_3)^2} \\
& + \tfrac{n_F(E_3)}{(E_3-E_1)(E_3-E_2)^2} \\
= & \sum_k \tfrac{n_F(E_1)}{(E_2-E_1)(E_1-E_3)(E_2-E_3)} \\
& - \tfrac{n_F(E_2)}{(E_2-E_1)(E_2-E_3)^2} + O(n''_F) \\
= & \sum_k \tfrac{n_F(E_1)(E_2-E_3)-n_F(E_2)(E_1-E_3)}{(E_2-E_1)(E_1-E_3)(E_2-E_3)^2} + O(n''_F) \\
= & \sum_k \tfrac{n_F(E_1)(E_2-E_1)+[n_F(E_1)-n_F(E_2)](E_1-E_3)}{(E_2-E_1)(E_1-E_3)(E_2-E_3)^2} + O(n''_F) \\
\sim & \sum_k O(n''_F) \sim 1/T^2
\end{aligned}$$

Hence we explicitly find that, taken in this form, the integrand is in fact non-divergent (apparently singular denominators are cancelled by the numerator at finite temperatures). Similarly,

$$\begin{aligned}
\Delta\Omega_B^{(4)} \sim & \sum_k \tfrac{n'_F(E_1)+n'_F(E_2)}{(E_2-E_1)^2} - 2\tfrac{n_F(E_1)-n_F(E_2)}{(E_1-E_2)^3} \\
= & \sum_k \tfrac{n'_F(E_1)(E_2-E_1)+n'_F(E_2)(E_2-E_1)-2n_F(E_1)+2n_F(E_2)}{(E_2-E_1)^3} \\
\approx & -\sum_k \tfrac{(2n'_F+n'''_F(\delta/2)^2)\delta-2(n'_F\delta+n'''_F(\delta/2)^3/3)}{\delta^3} = -n'''_F/6 \sim 1/T^3 \quad n_F \equiv n_F(E_m) \\
& \qquad \delta = E_1 - E_2 \\
& \qquad E_m = (E_1+E_2)/2
\end{aligned}$$

Note that this self-interaction (B term) is more singular as $T \to 0$ that the A term.