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INTRODUCTION

THE study of the superfluidity of helium II gave rise to the question of the properties of quantum fluids, that is, of systems of interacting particles in whose behavior quantum effects play a major part. The theory of superfluidity given by L. D. Landau was at the same time the first theory of a quantum fluid. It described the properties of so-called Bose fluids, i.e., quantum systems of interacting particles for which the excitations obey Bose statistics.

Besides such fluids, however, there also exist others, forming a much more numerous class; these are the so-called Fermi fluids, whose excitations have spin 1/2 and obey Fermi statistics. These include liquid helium 3, the electrons in metals, and possibly heavy nuclei, although to be sure this last case is a quite special one. Until very recently there was no theory of the Fermi fluid. The theoretical calculations were usually confined to an approach analogous to that used for Bose fluids; some sort of energy spectrum was postulated, and this was then substituted into the Fermi distribution formula.

In view of the fact that in metals the situation was decidedly complicated by the anisotropy of the spectra, it was hard to test the correctness of such an approach. As for liquid helium 3 at low temperatures, the experimental data made it very clear that in this case such an approach could not be completely successful.

A consistent theory of Fermi fluids was developed by L. D. Landau in 1956.[1] He showed that the situtation is very different from that in Bose fluids, since in Fermi fluids a very important part is played by the interaction of the excitations, so that the excitations in a Fermi fluid cannot be regarded in all cases as an ideal gas.

In the present article we expound the theory of the Fermi fluid in its application to the isotropic model and show how the Landau theory provides an explanation of the various properties of liquid helium 3. In an appendix we present a recently completed research of L. D. Landau, which shows how the basic propositions of the theory of the Fermi liquid follow from a microscopic treatment of the interaction, and we also present there the theory of a rarefied Fermi gas. In this case one can obtain the basic quantities of the theory of the Fermi fluid by an actual calculation.

Before passing to the exposition of the theory, we recall for convenience in later applications, the main properties of liquid helium 3. The helium isotope of atomic weight 3 liquefies at 3.2°K (critical point $T_c = 3.3°$, $p_c = 845$ mm Hg), and remains liquid down to the lowest temperatures at which it has been studied (~ 0.2°K) and at pressures up to 30 atmos. The explanation is the same as in the case of helium 4: the weakness of the interaction between the atoms, and the smallness of the atomic mass. Owing to the latter circumstance the deBroglie wavelength corresponding to the motion of helium 3 atoms at low temperatures can be larger than the distances between atoms; that is, the liquid becomes a quantum fluid.

Down to the lowest temperatures that have been used liquid helium 3 does not become a superfluid. Since according to the Landau theory every Bose liquid must possess superfluidity, while a Fermi liquid does not have this property, helium 3 must be regarded as a Fermi fluid. It must be stated that this fact is not trivial and does not follow automatically from the fact that helium 3 atoms have spin 1/2. A system of such atoms might have exclusively Bose excitations. Moreover, as we shall see later, such excitations actually exist, although they do not play any important part in determining the properties of helium 3. It is only the absence of superfluidity that enables us to conclude with assurance that helium 3 is a genuine Fermi fluid.

1. THE ENERGY OF THE EXCITATIONS

The possibility of describing an excited state of liquid helium 3 by means of a gas of quasi-particles and a corresponding distribution function in the energy scale is based on the fact that the interaction of the particles of a Fermi gas decreases rapidly as the temperature is lowered. In fact, it is clear from general considerations that the time between collisions is proportional to the square of the diffuseness of the Fermi distribution, that is, to the mean square of the temperature. At the same time the mean energy of the Fermi particles in the diffuse part of the distribution (measured from the limit energy) is itself proportional to the first power of the temperature, and it is these particles that determine the macroscopic properties of the system. Owing to this, the "energy uncertainty" arising from the collisions will, at low enough temperatures, be small in comparison with the mean energy itself. It can also be said that at sufficiently low temperatures the damping of the excitations, described by an imaginary term added to their energy, will be small in comparison with the excitation energy.

Two fundamental assumptions lie at the foundation of the Landau theory. The first is that the classification of the levels of the Fermi fluid corresponds to the classification of the levels of noninteracting atoms. This means that as one gradually turns on the interaction the atoms gradually go over into "quasi-particles", each of which has a definite energy. Thus the number of quasi-particles is equal to the number of atoms.

The second assumption is that the interaction of the quasi-particles can be taken into account by means of a self-consistent field of the surrounding particles, which manifests itself in the fact that the energy of the system is not the sum of the energies of the quasi-particles, but instead is a functional of their distribution function. The energy of the quasi-particles must be defined as the variational derivation of the energy density in momentum space, i.e.,

$$\delta E = \int \varepsilon \delta n \, d\tau, \tag{1.1}$$

where $d\tau = 2dp_x dp_y dp_z/(2\pi h)^3$. The factor 2 in $d\tau$ arises from the fact that the quasi-particles have spin 1/2.

In some cases it is necessary to include a spin dependence of the particle energy. Owing to the fact that the spin is a quantum-mechanical quantity, we must regard the distribution function in this case as a statistical operator, and replace (1.1) by a definition of ϵ in the form

$$\delta E = \frac{1}{2} \mathrm{Sp}_\sigma \int \varepsilon \, \delta n \, d\tau. \tag{1.2}$$

The definition of the particle energy by Eq. (1.2) has the consequence that the equilibrium distribution function is in fact the Fermi function. To prove this it is most convenient to use the expression for the entropy*

$$S = -\frac{1}{2} \mathrm{Sp}_\sigma \int \{n \ln n + (1-n) \ln (1-n)\} \, d\tau, \tag{1.3}$$

since this formula is of purely combinatorial origin. From the condition that the entropy be a maximum for constant number of particles and constant energy,

$$N = \frac{1}{2} \mathrm{Sp}_\sigma \int n \, d\tau = \mathrm{const}, \quad E = \mathrm{const}$$

we can find the distribution function by taking the variation with respect to n:

$$n(\varepsilon) = n_F(\varepsilon) = \frac{1}{e^{\frac{\varepsilon-\mu}{T}} + 1} \,. \tag{1.4}$$

The energy ϵ, being a functional of n, depends on the temperature. This dependence can be put in the following form. If we denote by $\epsilon_0(\mathbf{p}, \sigma)$ the equilibrium energy of the quasi-particles for T = 0, then for a small departure from equilibrium or for small values of T it will be given by the formula

$$\begin{aligned} \varepsilon = \varepsilon_0(\mathbf{p}, \sigma) + \delta\varepsilon(\mathbf{p}, \sigma) = \varepsilon_0(\mathbf{p}, \sigma) \\ + \frac{1}{2} \mathrm{Sp}_{\sigma'} \int f(\mathbf{p}, \sigma; \mathbf{p}', \sigma') \, \delta n(\mathbf{p}', \sigma') \, d\tau'. \end{aligned} \tag{1.5}$$

Here $\delta n = n - n_F\ (T = 0)$, and f is an operator depending on the momenta and spin operators of two particles. In view of its definition, as the second variational derivative of E with respect to δn, the function f must be symmetric under interchange of $\mathbf{p}, \sigma$ with $\mathbf{p}', \sigma'$. The function f is a very important quantity characterizing the Fermi fluid. As was shown by Landau, it is related to the forward scattering amplitude of two quasi-particles (cf. Appendix 2).

2. THE EFFECTIVE MASS

Because we are concerned here with very low temperatures, the energy ϵ_0 in the absence of an external field can be written in the form

$$\varepsilon_0 - \mu(0) = v(p - p_0), \tag{2.1}$$

where p_0 is the Fermi limiting momentum and v is the velocity at the Fermi limit. This velocity can be written in the form

$$v = \frac{p_0}{m^*}, \tag{2.2}$$

where m* is the effective mass. As has been shown by Landau, there is a definite relation between m*

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*Here and throughout we use energy units for the temperature, i.e., k = 1.

and f, which is a consequence of the Galilean principle of relativity.

According to this principle the momentum per unit volume must be equal to the mass flux of the fluid. Because the velocity for the quasi-particles can be defined as the derivative $\partial\epsilon/\partial\mathbf{p}$, and the number of quasi-particles is equal to the number of atoms, we get:

$$\frac{1}{2}\mathrm{Sp}_\sigma \int \mathbf{p}n\, d\tau = \frac{1}{2}\mathrm{Sp}_\sigma \int m\frac{\partial\varepsilon}{\partial\mathbf{p}} n\, d\tau, \tag{2.3}$$

where m is the mass of an atom.

Let us take the variation of this relation with respect to n:

$$\frac{1}{2}\mathrm{Sp}_\sigma \int \mathbf{p}\delta n\, d\tau = \frac{1}{2}\mathrm{Sp}_\sigma \int m\frac{\partial\varepsilon}{\partial\mathbf{p}}\delta n\, d\tau$$
$$+\frac{1}{2}\mathrm{Sp}_\sigma\mathrm{Sp}_{\sigma'}\int\int mn\frac{\partial}{\partial\mathbf{p}}f(\mathbf{p},\sigma,\mathbf{p}',\sigma')\delta n'\, d\tau\, d\tau'.$$

Interchanging **p**, **σ** and **p′**, **σ′** in the last term and using the fact that δn is arbitrary, we get:

$$\frac{\mathbf{p}}{m} = \frac{\partial\varepsilon}{\partial\mathbf{p}} + \frac{1}{2}\mathrm{Sp}_{\sigma'}\int\frac{\partial f}{\partial\mathbf{p}'}n'\, d\tau' = \frac{\partial\varepsilon}{\partial\mathbf{p}} - \frac{1}{2}\mathrm{Sp}_{\sigma'}\int f\frac{\partial n'}{\partial\mathbf{p}'}d\tau. \tag{2.4}$$

If the system is not in an external magnetic field, the relation (2.4) does not lose its generality when we take its trace with respect to the unprimed spin variable ($1/2\,\mathrm{Sp}\sigma$). In the case of equilibrium at $T = 0$, ϵ in Eq. (2.4) can be taken in the form (2.1), and $\partial n/\partial p$ can be replaced by $-\delta(p - p_0)$. We then find:

$$\frac{1}{m} = \frac{1}{m^*} + \frac{p_0}{2(2\pi\hbar)^3}\mathrm{Sp}_\sigma\mathrm{Sp}_{\sigma'}\int f(\chi)\cos\chi\, d\Omega, \tag{2.5}$$

where $f(\chi)$ is the value of the function f for $|\mathbf{p}| = |\mathbf{p}'| = p_0$. Naturally, apart from a small correction, this relation also holds for temperatures near $T = 0$ and for small departures from equilibrium.

3. THE HEAT CAPACITY AND ENTROPY

Here we shall assume no external magnetic field, so that the quantities will not depend on the spin. A knowledge of the energy spectrum makes it possible to determine the heat capacity of the Fermi fluid.[2] It can be found in the usual way, by differentiating the energy with respect to the temperature for a prescribed number of particles.

An important point here is that the change of the energy density is given by Eq. (1.1); owing to this we can write

$$C = \left(\frac{\partial E}{\partial T}\right)_N = \int\varepsilon\left(\frac{\partial n}{\partial T}\right)_N d\tau \cong \int\varepsilon_0\left(\frac{\partial n}{\partial T}\right)_N d\tau. \tag{3.1}$$

In the present case the replacement of ϵ by ϵ_0 is justified by the fact that whereas the main term in the heat capacity is linear in T the correction in Eq. (1.5) gives only cubic terms.

In fact, the δn in Eq. (1.5) can be put in the form

$$\delta n = n(T) - n(0) = \int_0^T \frac{\partial n}{\partial T}\, dT$$
$$= \int_0^T \frac{\partial n}{\partial\varepsilon}\left(-\frac{\varepsilon-\mu}{T} + \frac{\partial\varepsilon}{\partial T} - \frac{\partial\mu}{\partial T}\right)dT.$$

As is well known, at low temperatures the derivative of the Fermi function has the form

$$\frac{\partial n}{\partial\varepsilon} \approx -\delta(\varepsilon-\mu) - \frac{\pi^2}{6}T^2\frac{\partial^2\delta(\varepsilon-\mu)}{\partial\varepsilon^2}.$$

In view of this we have

$$\delta\varepsilon = \int f\delta n'\, d\tau' = \frac{\pi^2}{6}T^2\left[\frac{\partial}{\partial\varepsilon'}\left(f\frac{d\tau'}{d\varepsilon'}\right)\right]_{\varepsilon'=\mu}$$
$$-\int_0^T\left[f\left(\frac{\partial\varepsilon}{\partial T} - \frac{\partial\mu}{\partial T}\right)\frac{d\tau'}{d\varepsilon'}\right]_{\varepsilon'=\mu}dT. \tag{3.2}$$

The derivative $\partial\mu/\partial T$ is equal to the entropy, which, as we shall see, is linear in T. As for the term in $\partial\epsilon'/\partial T$, its order of magnitude can be established by Eq. (3.2). Differentiating this formula with respect to T, we find without difficulty that $\partial\epsilon/\partial T$ is of the first order in T, and consequently $\delta\epsilon$ is of the second order.

Because of the replacement ϵ by ϵ_0 the energy can be calculated simply from the integral

$$E = \int\varepsilon_0 n\, d\tau \quad \text{with} \quad N = \int n\, d\tau = \text{const},$$

where, again to within an error of cubic terms in T, we can replace ϵ by ϵ_0 in the functions n appearing in the integrals for E and N. After this the calculation of the integrals does not differ from that for the usual case of a Fermi gas.

In this way we find:

$$c = \frac{C}{N} = \gamma T; \quad \gamma = \frac{\pi^2}{3N}\left(\frac{d\tau}{d\varepsilon}\right)_\mu = \left(\frac{\pi}{3N}\right)^{\frac{2}{3}}\frac{m^*}{\hbar^2}. \tag{3.3}$$

Because of the linear dependence on the temperature the entropy is equal to the heat capacity.

By comparing Eq. (3.3) with the experimental data on the entropy of liquid He^3 at low temperatures we can find the parameter γ, and consequently can also determine the effective mass. Unfortunately, at present the experimental curve has only been found as far as the beginning of the linear part.[3] From these data we can obtain only an approximate value of γ, which turns out to be about 3 cal mole^{-1} deg^{-2}. Because of the equality of the number of quasi-particles to the number of particles, we can find the Fermi limit momentum from the density of liquid He^3 ($\rho = 0.078$ g/cm^3). In this way we get:

$$m^* = 1.43 m_{He^3}, \quad \frac{p_0}{\hbar} = 0.76 \cdot 10^8 \text{ cm}^{-1}. \tag{3.4}$$

4. THE MAGNETIC SUSCEPTIBILITY

Because particles possessing spin necessarily have a magnetic moment, the Fermi fluid must be magnetic. In the isotropic case there are only two possibilities. The fluid can be paramagnetic or ferromagnetic. Antiferromagnetism in an isotropic Fermi fluid is impossible. We present here only the case of the paramagnetic Fermi liquid, which corresponds to liquid He^3. A calculation of the susceptibility of such a liquid was given by Landau in reference 1 (cf. also reference 2).

When the system is in a magnetic field H, the particles naturally have an additional energy depending on their spins. If they were free particles, the change of the energy would be $-\beta(\sigma \cdot H)$. In the Fermi fluid, however, we have to take into account also the fact that there is a change in the distribution function. Thus we get

$$\delta\varepsilon = -\beta(\boldsymbol{\sigma}\cdot\mathbf{H}) + \frac{1}{2}\mathrm{Sp}_{\sigma'}\int f\delta n' d\tau'. \tag{4.1}$$

In the present case the spin dependence of the function f is important. This dependence can be written in the general case in the form

$$f(\mathbf{p}, \boldsymbol{\sigma};\ \mathbf{p}', \boldsymbol{\sigma}') = f(\mathbf{p}, \mathbf{p}') + \zeta_{ik}(\mathbf{p}, \mathbf{p}')\sigma_i\sigma'_k. \tag{4.2}$$

It is natural to assume, however, that the spin interaction is mainly of exchange origin. In this case f takes the form

$$f(\mathbf{p}, \boldsymbol{\sigma};\ \mathbf{p}', \boldsymbol{\sigma}') = f(\mathbf{p}, \mathbf{p}') + \zeta(\mathbf{p}, \mathbf{p}')\,\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'. \tag{4.3}$$

The change of the distribution function appearing in Eq. (4.1), δn, is due only to the change of the energy ϵ, since the chemical potential changes only in the second order in H. Because of this we can set $\delta n = \partial n/\partial\epsilon\ \delta\epsilon$ in Eq. (4.1). This gives

$$\delta\varepsilon = -\beta(\boldsymbol{\sigma}\cdot\mathbf{H}) + \frac{1}{2}\mathrm{Sp}_{\sigma'}\int f\frac{\partial n'}{\partial\varepsilon'}\delta\varepsilon' d\tau'. \tag{4.4}$$

This is the equation for $\delta\epsilon$, which can be assumed to have the form

$$\delta\varepsilon = -\xi(\boldsymbol{\sigma}\cdot\mathbf{H}). \tag{4.5}$$

Substituting this into Eq. (4.4), we get the equation

$$\xi = \beta + \frac{1}{4}\int \zeta \frac{\partial n'}{\partial\varepsilon'}\xi' d\tau'. \tag{4.6}$$

In view of the fact that $\partial n'/\partial\epsilon' \cong -\delta(\epsilon' - \mu)$, the integration is taken over the surface of the Fermi sphere. We shall see below that only the value of ξ taken on the surface of the Fermi sphere is of importance for the magnetic susceptibility. In this case both the arguments of ζ (p, p′) have the absolute value p_0, and ζ depends only on the angle between them. Denoting $\int\zeta(\theta)\, d\Omega/4\pi$ by $\bar{\zeta}$, we get from Eq. (4.6):

$$\xi = \frac{\beta}{1 + \frac{1}{4}\bar{\zeta}\left(\frac{d\tau}{d\varepsilon}\right)_\mu}. \tag{4.7}$$

The susceptibility is found from the formula

$$\chi = \frac{\partial}{\partial \mathbf{H}}\frac{1}{2}\mathrm{Sp}\int n\beta\boldsymbol{\sigma}\, d\tau.$$

The value of this expression is of course determined by just the spin-dependent change of n. Thus we find:

$$\chi = \frac{1}{4}\beta\xi\left(\frac{d\tau}{d\varepsilon}\right)_\mu = \frac{\frac{1}{4}\beta^2\left(\frac{d\tau}{d\varepsilon}\right)_\mu}{1 + \frac{1}{4}\bar{\zeta}\left(\frac{d\tau}{d\varepsilon}\right)_\mu}. \tag{4.8}$$

The quantity $(\partial\tau/\partial\epsilon)_\mu$ can be expressed in terms of the coefficient γ in the linear law of the heat capacity; this gives

$$\frac{1}{\chi} = \beta^{-2}\left(\bar{\zeta} + \frac{4\pi^2}{3\gamma N}\right). \tag{4.9}$$

Equation (4.9) can be compared with the experimental data on the magnetic susceptibility of He^3;[4] this shows that $\bar{\zeta}$ is negative and is of magnitude 0.85 times the second term. Thus the exchange interaction has a decided effect on the magnitude of the magnetic susceptibility of liquid He^3. The sign of the effect is such that the exchange interaction facilitates parallel orientation of the spins. It does not lead to ferromagnetism, however, since the Fermi tendency toward an antiparallel arrangement of the spins prevails. It is possible that at higher pressures the parameters change in such a way that ferromagnetism becomes possible. There are no experimental data up to the present, however, to support such a conclusion.

5. THE KINETIC EQUATION

The kinetic equation for the distribution function in the absence of a magnetic field has the usual form

$$\frac{\partial n}{\partial t} + \frac{\partial n}{\partial \mathbf{r}}\cdot\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\cdot\frac{\partial \varepsilon}{\partial \mathbf{r}} = I(n), \tag{5.1}$$

where I(n) is the collision integral. Here, however, we have to take into account the fact that the energy ϵ is a functional of the distribution function, and thus also depends on the coordinates. By means of this kinetic equation Landau[1] obtained expressions for the fluxes of energy and momentum.

To find the momentum flux one must multiply (5.1) by p_i and integrate over the phase space. Conservation of momentum in the collisions makes the integral $\int p_i I(n) d\tau$ vanish, and we have the equation

$$\frac{\partial}{\partial t}\int p_i n\, d\tau + \int p_i\left(\frac{\partial n}{\partial x_k}\cdot\frac{\partial \varepsilon}{\partial p_k} - \frac{\partial n}{\partial p_k}\cdot\frac{\partial \varepsilon}{\partial x_k}\right) d\tau = 0.$$

This expression in parentheses in the second integral can be written

$$\frac{\partial}{\partial x_k}\left(n\frac{\partial \varepsilon}{\partial p_k}\right) - \frac{\partial}{\partial p_k}\left(n\frac{\partial \varepsilon}{\partial x_k}\right).$$

Substituting this into the integral and integrating the second term by parts, we get:

$$\frac{\partial}{\partial x_k}\int p_i\frac{\partial \varepsilon}{\partial p_k} n\, d\tau + \int n\frac{\partial \varepsilon}{\partial x_i} d\tau.$$

The last term can be put in the form

$$\frac{\partial}{\partial x_i}\int n\varepsilon\, d\tau - \int \varepsilon\frac{\partial n}{\partial x_i} d\tau.$$

But in view of the fact that $\delta E = \int \epsilon \delta n d\tau$, the last term here is equal to $\partial E/\partial x_i$. Thus we get the equation

$$\frac{\partial}{\partial t}\int p_i n\, d\tau + \frac{\partial \Pi_{ik}}{\partial x_k} = 0. \tag{5.2}$$

This equation expresses the law of conservation of momentum. The tensor Π_{ik} is the flux of momentum. It is given by

$$\Pi_{ik} = \int p_i\frac{\partial \varepsilon}{\partial p_k} n\, d\tau + \delta_{ik}\left[\mathrm{Sp}\int n\varepsilon\, d\tau - E\right]. \tag{5.3}$$

In a similar way we also get the law of conservation of energy. To do this we multiply the kinetic equation by ϵ and integrate with respect to $d\tau$. Because of the conservation of energy in the collisions the integral $\int \epsilon I(n) d\tau$ vanishes, and we get:

$$\int \varepsilon\frac{\partial n}{\partial t} d\tau + \int \varepsilon\left(\frac{\partial n}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} - \frac{\partial n}{\partial \mathbf{p}}\cdot\frac{\partial \varepsilon}{\partial \mathbf{r}}\right) d\tau = 0.$$

Writing the term in $\partial n/\partial \mathbf{r}$ in the form

$$\frac{\partial}{\partial \mathbf{r}}\int n\varepsilon\frac{\partial \varepsilon}{\partial \mathbf{p}} d\tau - \int n\frac{\partial \varepsilon}{\partial \mathbf{r}}\frac{\partial \varepsilon}{\partial \mathbf{p}} d\tau - \int n\frac{\partial^2 \varepsilon}{\partial \mathbf{r}\,\partial \mathbf{p}}\varepsilon\, d\tau$$

and integrating by parts with respect to $\mathbf{p}$ in the term in $\partial n/\partial \mathbf{p}$, we get the law of conservation of energy

$$\frac{\partial E}{\partial t} = \int \varepsilon\frac{\partial n}{\partial t} d\tau = -\operatorname{div}\mathbf{Q}, \tag{5.4}$$

where the flux of energy is given by

$$\mathbf{Q} = \int n\varepsilon\frac{\partial \varepsilon}{\partial \mathbf{p}} d\tau. \tag{5.5}$$

6. THE VISCOSITY

From (5.1) and the expressions (5.2) for the momentum flux and (5.5) for the energy flux we can determine the coefficients of viscosity and thermal conductivity of the Fermi fluid.[5]

We begin by finding the viscosity. To do this, suppose that there occurs a motion in the liquid with a certain velocity $\mathbf{u}$ which is a slowly varying function of position. In this case the distribution function will differ only slightly from the equilibrium value:

$$n = n_0 + \delta n, \tag{6.1}$$

where

$$n_0 = \frac{1}{e^{\frac{\varepsilon_0 - \mathbf{p}\cdot\mathbf{u} - \mu}{T}} + 1}; \quad |\delta n| \ll n_0. \tag{6.2}$$

The quantity δn is found from the kinetic equation (5.1). As usual, we must insert the function n_0 in the left member of the kinetic equation. Furthermore we shall assume that at the point considered $\mathbf{u} = 0$. Substituting Eq. (6.2) into (5.1) we find:

$$\frac{\partial n_0}{\partial t} - \frac{1}{3}\frac{\partial n_0}{\partial \varepsilon}\mathbf{p}\frac{\partial \varepsilon}{\partial \mathbf{p}}\operatorname{div}\mathbf{u} - \frac{1}{2}\frac{\partial n_0}{\partial \varepsilon}\left(p_i\frac{\partial \varepsilon}{\partial p_k} - \frac{1}{3}p_l\frac{\partial \varepsilon}{\partial p_l}\delta_{ik}\right) \times\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\delta_{ik}\frac{\partial u_l}{\partial x_l}\right) = I(n). \tag{6.3}$$

We now transform $\partial n_0/\partial t$ and show that this expression also does not depend on the term in f in Eq. (1.5). According to Eq. (6.2) we can write

$$\delta n_0 = \frac{\partial n_0}{\partial \varepsilon_0}\left(\delta\varepsilon_0 - \delta\mu - \frac{\varepsilon_0 - \mu}{T}\delta T - \mathbf{p}\cdot\delta\mathbf{u}\right). \tag{6.4}$$

Since the derivative $\partial n_0/\partial \epsilon_0$ in Eq. (6.4) is different from zero in a small neighborhood of the point $\epsilon_0 = \mu$, and is a rapidly changing function in this region, we can suppose the quantities in the parentheses evaluated at that point (the error will be of the relative order $(T/\mu)^2$). The variations $\delta\epsilon_0$ and $\delta\mu$ are arbitrary and are by no means equal to each other. On the other hand, the distribution function is normalized by the relation

$$\int n_0 d\tau = N,$$

where N is the number of atoms in unit volume. Taking the variation of this relation, we find:

$$\begin{aligned}\delta N &= \int \frac{\partial n_0}{\partial \varepsilon_0}\left(\delta\varepsilon_0 - \delta\mu - \frac{\varepsilon_0 - \mu}{T}\delta T - \mathbf{p}\cdot\delta\mathbf{u}\right) d\tau \\ &\approx -\left[(\delta\varepsilon_0 - \delta\mu - \mathbf{p}\cdot\delta\mathbf{u})\frac{d\tau}{d\varepsilon_0}\right]_{\varepsilon_0 = \mu},\end{aligned} \tag{6.5}$$

since $\int (\partial n_0/\partial \epsilon_0) d\epsilon_0 = -1$. Comparing (6.5) and (6.4), we get:

$$\frac{\partial n_0}{\partial t} = -\frac{\partial N}{\partial t}\cdot\frac{\partial n_0}{\partial \varepsilon}\cdot\left(\frac{d\varepsilon_0}{d\tau}\right)_{\varepsilon_0 = \mu}.$$

The number of atoms N satisfies the equation of continuity

$$\frac{\partial N}{\partial t} + N\operatorname{div}\mathbf{u} = 0. \tag{6.6}$$

Thus the term in $\partial n_0/\partial t$ gives an addition to the term in div **u**, which as a result now takes the form

$$-\frac{\partial n_0}{\partial \varepsilon}\left(\frac{1}{3}\,\mathbf{p}\,\frac{\partial \varepsilon_0}{\partial \mathbf{p}} - N\left(\frac{d\varepsilon_0}{d\tau}\right)_{\varepsilon_0=\mu}\right)\operatorname{div}\mathbf{u}. \tag{6.7}$$

Using Eq. (2.1) and the equality of the number of excitations to the number of particles, which makes $p_0/h = (3\pi^2 N)^{1/3}$, we find that the expression just written vanishes. Thus apart from small corrections of the order $(T/\mu)^2$ there are no terms in div **u** in the left member of the kinetic equation.

The right member of Eq. (5.1) is the collision integral

$$I(n) = -\int w\,[n_1 n_2 (1 - n_1')(1 - n_2') - (1 - n_1)(1 - n_2)\,n_1' n_2']$$

$$\times\,\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\,\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')\,d\tau_2 d\tau_1' d\mathbf{p}_2'. \tag{6.8}$$

Here we must keep in mind that the δ function of the energies has as its arguments the actual energies, i. e., the energies as given by Eq. (1.5). Owing to this, substitution of the function $n_0(\epsilon_0)$ into the collision integral does not make it vanish. The integral vanishes only when we substitute in it the true equilibrium function, i. e., $n_0(\epsilon)$. The total distribution function can be written

$$n = n_0(\varepsilon) - \frac{\partial n_0}{\partial \varepsilon}\int f\,\delta n\,d\tau + \delta n. \tag{6.9}$$

We introduce the notation

$$\delta n = -\frac{n_0(1 - n_0)}{T}\,\nu = \frac{\partial n_0}{\partial \varepsilon_0}\,\nu. \tag{6.10}$$

Substituting this in Eq. (6.9), we find that the term added to the equilibrium distribution function also has the form (6.10). Instead of ν however, it contains the quantity

$$\psi = \nu + \left(\frac{d\tau}{d\varepsilon_0}\right)_\mu \int f(\chi)\,\nu\,\frac{d\Omega}{4\pi}. \tag{6.11}$$

Substitution of Eq. (6.9) in the collision integral, with neglect of terms quadratic in ψ, gives:

$$I(n) = \frac{1}{T}\int w n_{01} n_{02} (1 - n_{01}')(1 - n_{02}')(\psi_1 + \psi_2 - \psi_1' - \psi_2')$$

$$\times\,\delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\,\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')\,d\tau_2 d\tau_1' d\mathbf{p}_2'. \tag{6.12}$$

In general the collision probability w depends on all four momenta. The momenta of importance here, however, are those with magnitudes close to the limit value on the Fermi surface. We can therefore assume that w depends only on θ, the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, and φ, the angle between the planes $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_1', \mathbf{p}_2')$.

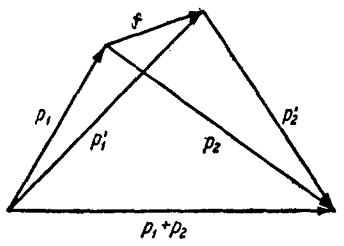

FIG. 1

Let us make use of the fact that the momenta of the particles in the main region of the integration differ little in magnitude from the limit momentum p_0. If we rotate the plane of the vectors $(\mathbf{p}_1', \mathbf{p}_2')$ through the angle φ relative to the axis in the direction $\mathbf{p}_1 + \mathbf{p}_2$, so that this plane coincides with the plane $(\mathbf{p}_1, \mathbf{p}_2)$, we get the diagram shown in Fig. 1. It is clear that the vector **f** will be small in magnitude and that the angles between all the momenta and the axis $\mathbf{p}_1 + \mathbf{p}_2$ will be approximately equal to $\theta/2$ or $-\theta/2$. Thus we get

$$p_1' \approx p_1 + f_z \cos\frac{\theta}{2} + f_r \sin\frac{\theta}{2}, \quad p_2' \approx p_2 - f_z \cos\frac{\theta}{2} + f_r \sin\frac{\theta}{2}, \tag{6.13}$$

where f_z is the component of **f** along the axis $\mathbf{p}_1 + \mathbf{p}_2$ and f_r is the perpendicular component. We get rid of the δ function by integrating over $d\mathbf{p}_2'$, and replace the integral over $d\tau_1'$ by that over $df_r df_z d\varphi$, introducing a system of cylindrical coordinates with axis along $\mathbf{p}_1 + \mathbf{p}_2$:

$$d\tau_1' = 2p_0 \sin\frac{\theta}{2}\,df_r df_z d\varphi\,\frac{1}{(2\pi\hbar)^3}. \tag{6.14}$$

We now introduce the following notation:

$$x = \frac{\varepsilon_1' - \mu}{T}, \quad t = \frac{\varepsilon_1 - \mu}{T}, \quad \varkappa = \frac{\mu}{T}, \quad y = \frac{\varepsilon_2' - \mu}{T}; \tag{6.15}$$

with the values of ϵ_i from Eq. (2.1). The change from the variables f_r and f_z to x and y is accomplished very easily by means of Eq. (6.13). Then we get:

$$d\tau_1' \int d\tau_2\,\delta(\varepsilon) = \frac{m^{*3} T^2 dx\,dy \left(\frac{d\Omega}{2\pi}\right)\left(\frac{d\varphi_2}{2\pi}\right)}{8\pi^4 \hbar^6 \cos\left(\frac{\theta}{2}\right)}, \tag{6.16}$$

where we have denoted the angular differential $\sin\theta\,d\theta\,d\varphi$ by $d\Omega$, which involves the arguments of $w(\theta, \varphi)$. We note that because of the indistinguishibility of the particles the angle φ varies only from zero to π.

From considerations of symmetry it is clear that ψ must have the form

$$\psi = \frac{1}{2}\,q(t)\left(p_i \frac{\partial \varepsilon}{\partial p_k} - \frac{1}{3}\,p_l \frac{\partial \varepsilon}{\partial p_l}\,\delta_{ik}\right)\cdot\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} - \frac{2}{3}\,\delta_{ik}\frac{\partial u_m}{\partial x_m}\right). \tag{6.17}$$

When this expression is substituted in the collision integral the second factor can be transformed by means of the addition theorem for spherical harmonics. After the integration over the angle φ_2 only the first terms remain, i.e.,

$$P_2(\theta_1 + \theta_2) \longrightarrow P_2(\theta_1)\,P_2(\theta_2), \text{ where } P_2(\theta) = \frac{3}{2}\cos^2\theta - 1.$$

As the result we get from Eq. (6.3) the following equation for q:

$$n_0(t)\,[1-n_0(t)] = \frac{m^{*3}T^2}{8\pi^4\hbar^6\cos\frac{\theta}{2}} \int \frac{d\Omega}{2\pi} \underset{x+y>t-\varkappa}{\int_{-\varkappa}^{\infty} dx \int_{-\varkappa}^{\infty}} dy\, w(\theta,\varphi)$$

$$\times\, n_0(t)\, n_0(x+y-t)\,[1-n_0(x)]\,[1-n_0(y)] \qquad (6.18)$$

$$\times\, [q(t) + q(x+y-t)\,P_2(\theta) - q(x)\,P_2(\theta') - q(y)\,P_2(\theta_2')].$$

In view of the fact that in the range of temperatures we are considering only values of x, y, and t much smaller than $\varkappa$ are of importance, we can take the lower limit of the integration in Eq. (6.12) to be $-\infty$. Under these conditions, assuming that q is a symmetric function (as is confirmed by the results) we easily verify that all the terms with different q's can be reduced to the same form as regards the dependence on x and y. Thus the bracketed expression takes the simple form

$$q(t) + q(x)\,[P_2(\theta) - P_2(\theta_1') - P_2(\theta_2')].$$

The change from the angles θ_1' and θ_2' to the angles θ and φ is easily obtained from Fig. 1. Integrating over the variable y, on which q no longer depends, after some manipulation we get the following equation:

$$\frac{8\pi^4\hbar^6}{m^{*3}T^2} = \overline{\frac{w(\theta,\varphi)}{\cos\frac{\theta}{2}} \cdot \left[\frac{3}{4}(1-\cos\theta)^2\sin^2\varphi - 1)\right]}$$

$$\times \left\{ \int_0^{\infty} \frac{dx\, x\,[q(x+t)+q(x-t)]}{e^x-1} + 2\int_0^{\infty} \frac{dx\, xq(x)}{e^x+1} + \int_0^{t} dx \cdot xq(x-t) \right\}$$

$$+ \overline{\frac{w(\theta,\varphi)}{\cos\frac{\theta}{2}}}\, q(t) \cdot \frac{\pi^2+t^2}{2}, \qquad (6.19)$$

where the bar denotes averages over the solid angle.

The equation that has been obtained is complicated, but analysis shows that for arbitrary assumptions about the form of $w(\theta, \varphi)$ the error in the coefficient of viscosity will be smaller than 10 percent if we simply assume that the values we need satisfy $t^2 \ll \pi^2$. Then q is a constant, given by

$$q = \frac{64\pi^2\hbar^6}{3m^{*3}T^2} \left[\overline{\frac{w(\theta,\varphi)}{\cos\left(\frac{\theta}{2}\right)} (1-\cos\theta)^2 \sin^2\varphi} \right]^{-1}. \qquad (6.20)$$

The flux of momentum is given by (5.3). Inserting in it $n = n_0 + \delta n$ and Eq. (6.10), and using the fact that the energy is also a functional of the distribution function, we get:

$$\Pi_{ik} = \int p_i \frac{\partial\varepsilon(p)}{\partial p_k} \left[\delta n - \frac{\partial n_0}{\partial\varepsilon_0} \int f(\mathbf{p}, \mathbf{p}')\, \delta n'\, d\tau' \right] d\tau$$

$$= \int p_i \frac{\partial\varepsilon(p)}{\partial p_k} \frac{\partial n_0}{\partial\varepsilon_0} \left[\nu + \int f\nu \left(\frac{d\tau'}{d\varepsilon_0'}\right)_{\varepsilon_0'=\mu} \frac{d\Omega'}{4\pi} \right] d\tau \qquad (6.21)$$

$$= \int p_i \frac{\partial\varepsilon(p)}{\partial p_k} \frac{\partial n_0}{\partial\varepsilon_0}\, \psi\, d\tau.$$

Thus it is seen that the expression for the momentum flux involves the same function ψ that appears in the kinetic equation. Consequently, the function f does not enter the expression for the momentum flux, and it has the same form as for a Fermi gas of particles with the mass m* and the scattering law described by the function $w(\theta, \varphi)$.*

Substituting Eqs. (6.17) and (6.20) in Eq. (6.21) and defining the coefficient of viscosity as the proportionality coefficient between Π_{ik} and $-(\partial u_i/\partial x_k + \partial u_k/\partial x_i - 2/3\; \partial u_l/\partial x_l\, \delta_{ik})$, we find

$$\eta = \frac{64}{45}\, T^{-2} \frac{\hbar^3 p_0^5}{m^{*4}} \left[\overline{\frac{w(\theta,p)}{\cos\frac{\theta}{2}} (1-\cos\theta)^2 \sin^2\psi} \right]^{-1}. \qquad (6.22)$$

The viscosity is proportional to T^{-2}. This dependence was earlier predicted by Pomeranchuk[6] on the basis of qualitative considerations. As for the numerical values of the viscosity, they depend on the definite form of the averaged function $w(\theta, \varphi)$ and therefore cannot be found precisely. But Eq. (6.22) makes it possible to determine the order of magnitude of η. To do this we make use of the fact, which will be proved in Appendix 2, that the function f is of the order of magnitude of the scattering amplitude of the quasi-particles. Consequently $w \sim (2\pi/\hbar) f^2$, and the value of f can be determined from experimental data on the speed of sound (cf. Sec. 8). Using the numerical values of m* and p_0 from Eq. (3.4), we get for He^3

$$\eta = \frac{\alpha}{T^2}\ (\alpha \sim 10^{-6} \div 10^{-5}\ \text{poise};\ T\ \text{в}\ ^\circ\text{K}). \qquad (6.23)$$

This order of magnitude corresponds to the experimental results found by K. N. Zinov'eva.[7] Because her measurements went down only to 0.35°K, it is indeed impossible to check the temperature dependence. Nevertheless the experimental results show an increase of the viscosity with decreasing temperature.

As already noted early in this section, apart from terms of order $(T/\mu)^2$ there is no part pro-

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*In reference 5 the second term in the formula (6.9) was not included, and incorrect formulas were thus obtained for the viscosity and thermal conductivity coefficients. The correct values can be obtained by taking f = 0 in the expressions (26) and (30) of reference 5.

portional to div **u** on the left side of the kinetic equation. This means that the coefficient of second (dilatational) viscosity is smaller than η by a factor $(T/\mu)^2$ and plays no part in physical effects (for example, the absorption of sound).

7. THE THERMAL CONDUCTIVITY

We now go on to the calculation of the thermal conductivity.[5] Assuming a small temperature gradiant, we get in the left side of the kinetic equation:

$$-\frac{\partial n_0}{\partial \varepsilon_0}\left(\frac{\varepsilon_0-\mu}{T}-s\right)\frac{\partial \varepsilon_0}{\partial \mathbf{p}}\nabla T. \tag{7.1}$$

At the beginning the transformations of the collision integral are the same as those made in the preceding section [Eqs. (6.8) to (6.16)]. In the present case it is reasonable to try to find the function ψ in the form

$$\psi = q(t)\frac{\partial \varepsilon_0}{\partial \mathbf{p}}\nabla T. \tag{7.2}$$

Instead of Eq. (6.8) we now get a similar equation, but with $\cos\theta$ instead of $P_2(\cos\theta)$ in the collision integral and the factor $(\epsilon_0-\mu)/T - s$ in the left member. The presence of such a factor shows that the unknown function must contain parts both symmetric and antisymmetric in t, namely:

$$q(t) = q_s(t) + q_a(t). \tag{7.3}$$

Substitution in the collision integral gives the two equations

$$\begin{aligned}\frac{8\pi^4\hbar^6}{m^{*3}T^2}s = \frac{\overline{w(\theta,\varphi)}}{\cos\frac{\theta}{2}}\Big[\, q_s(t)\frac{\pi^2+t^2}{2} \\ -\int_0^\infty dx\,\frac{x[q_s(x+t)+q_s(x-t)]}{e^x-1} \\ -2\int_0^\infty \frac{dx\, xq_s(x)}{e^x+1} - \int dx\, xq_s(x-t)\Big],\end{aligned} \tag{7.4}$$

$$\begin{aligned}\frac{8\pi^4\hbar^6}{m^{*3}T^2} = \frac{\overline{w(\theta,\varphi)(1+2\cos\theta)}}{\cos\frac{\theta}{2}}\Big[\int_0^\infty dx\,\frac{x[q_a(x-t)-q_a(x+t)]}{e^x-1} \\ +2t\int_0^\infty \frac{dx\, q_a(x)}{e^x+1} + \int_0^t dx\cdot xq_a(x-t)\Big] + \frac{\overline{w(\theta,\varphi)}}{\cos\frac{\theta}{2}}q_a(t)\frac{\pi^2+t^2}{2}.\end{aligned} \tag{7.5}$$

In the present case, however, the kinetic equation does not completely determine the solution of the problem. One must use along with it a supplementary condition, which expresses the conservation of the flux of matter. This is

$$\begin{aligned}\delta\int n\frac{\partial \varepsilon}{\partial \mathbf{p}}d\tau = \int \delta n\frac{\partial \varepsilon_0}{\partial \mathbf{p}}d\tau + \int n_0\frac{\partial}{\partial \mathbf{p}}\Big[\int f\,\delta n'\,d\tau'\Big]d\tau \\ = \int \frac{\partial \varepsilon_0}{\partial \mathbf{p}}\Big[\delta n - \frac{\partial n_0}{\partial \varepsilon_0}\int f\,\delta n' d\tau'\Big]d\tau = 0.\end{aligned} \tag{7.6}$$

Here we have performed an integration by parts and have replaced ϵ by ϵ_0 in the integrals that already contain δn. Substituting the expressions (6.10), (6.11), and (7.2) in Eq. (7.6), we get

$$\int \frac{\partial n_0}{\partial \varepsilon_0}\left(\frac{\partial \varepsilon_0}{\partial \mathbf{p}}\right)^2 q(t)\,d\tau = 0. \tag{7.7}$$

To find $q_a(t)$ we have only to solve (7.5) in a way like that used in the treatment of the viscosity. The situation is different for the even part $q_s'(t)$. First, we must note that q_s = constant makes the right member of Eq. (7.4) vanish identically. Therefore the constant term $q_s(0)$ is to be determined not from this equation, but from the condition (7.7). Furthermore it is easy to show that the subsequent terms in $q_s(t)$, namely $a_2t^4 + a_4t^4 + \ldots$, make no contribution to the thermal conductivity. In fact, the occurrence of such a term as a_mt^{2m}, for example, must change the constant term in $q_s(t)$ by a quantity a_m^0 which is such that

$$\int \frac{\partial n_0}{\partial \varepsilon_0}\left(\frac{\partial \varepsilon}{\partial \mathbf{p}}\right)^2[a_mt^{2m}+a_m^0]\,d\tau = 0.$$

From this condition we get:

$$a_m^0 = -a_m(2m)!\,R_m;\quad R_m = \int_0^\infty \frac{z^{2m-1}}{e^z+1}dz.$$

To find the energy flux we have to calculate integrals of the form

$$\int \frac{\partial n_0}{\partial \varepsilon_0}F(\varepsilon_0)(a_mt^{2m}+a_m^0)\,d\tau$$

$$= \left(F\frac{d\tau}{d\varepsilon_0}\right)_{\varepsilon_0=\mu}[a_m(2m)!\,R_m + a_m^0] = 0.$$

Thus we get the result that to find the thermal conductivity coefficient we need only solve (7.5) and then choose the constant term so as to satisfy the condition (7.7). As in the case of the viscosity, it is enough for practical purposes to find the solution on the assumption $t^2 \ll \pi^2$. We then get:

$$q = \frac{24\pi^2h^6}{m^{*3}T^2}\left(t - \frac{\pi^2T_m^*}{p_0^2}\right)\left[\frac{\overline{w(\theta,\varphi)(1-\cos\theta)}}{\cos\frac{\theta}{2}}\right]^{-1}. \tag{7.8}$$

The energy flux is given by Eq. (5.5). Substituting Eq. (6.1) in that formula, we have:

$$Q = \int \varepsilon_0(\mathbf{p})\frac{\partial \varepsilon_0(\mathbf{p})}{\partial \mathbf{p}}\delta n\,d\tau - \int \varepsilon_0(\mathbf{p})\frac{\partial \varepsilon_0(\mathbf{p})}{\partial \mathbf{p}}\cdot\frac{\partial n}{\partial \varepsilon_0}\int f\delta n'\,d\tau'\,d\tau. \tag{7.9}$$

Substituting Eqs. (6.10), (6.11), (7.2), and (7.8) we find the value of the thermal conductivity coefficient

$$\varkappa = \frac{8}{3}\frac{\pi^2\hbar^3 p_0^3}{m^{*4}T}\cdot\left[\overline{\frac{w(\theta,\varphi)(1-\cos\theta)}{\cos\frac{\theta}{2}}}\right]^{-1}. \tag{7.10}$$

As in the case of the viscosity, the function f does not appear in the coefficient κ.

The temperature dependence of the thermal conductivity coefficient is given by the T^{-1} law, which also agrees with the qualitative prediction of Pomeranchuk.

Estimating the numerical value of κ for He^3 in the same way as we did that of the viscosity, we get the relation

$$\varkappa = \frac{\beta}{T}\,(\beta \sim 10^2 \div 10^3 \text{ erg cm}^{-1}\text{ sec}^{-1}\text{ deg}^{-1}\text{ ; T in °K}). \tag{7.11}$$

In any case the expressions obtained for the viscosity and thermal conductivity coefficients cease to be valid for temperatures $T \sim \mu$. In addition to this, however, there is a limitation that shifts the region of applicability of the theory toward considerably lower temperatures. This is that the range of excitation energies with which we are concerned, of the order of T, must be much larger than the quantum uncertainty in the energy that arises owing to the collisions, i.e.,

$$\tau \gg \frac{\hbar}{T}, \tag{7.12}$$

where τ is the time between collisions. We note that the condition (7.12) is required not only for the calculation of the kinetic coefficients as it was done above, but also for the validity of the entire theory of the Fermi fluid. For He^3 the quantity τ can be determined from the experimental values of the viscosity[7] [cf. Eq. (9.12)]. If we use the point at the very lowest temperature, i.e., $\eta = 4.8 \times 10^{-5}$ poise at T = 0.35°K, we get from Eq. (7.12)

$$T \ll 0{,}35°\,\text{K}. \tag{7.13}$$

It is clear, however, that this value is too high, since at T = 0.35°K the viscosity is still not proportional to T^{-2}, but follows a weaker law of variation.

8. THE PROPAGATION OF SOUND

The propagation of sound in a Fermi fluid has a number of quite specific peculiarities. If we consider sound of a prescribed frequency, then at not too low temperatures its propagation takes place according to the laws of ordinary hydrodynamics. In this case the damping of the sound is proportional to τ. When the temperature is lowered the collision time will increase proportional to T^{-2}, and finally, at temperatures such that $\omega\tau \sim 1$, the sound ceases to be propagated at all. It turns out, however, that when the temperature is lowered still further the possibility of propagation of sound reappears. The velocity will then be different, and the sound is no longer simply a wave of compression and rarefaction. This phenomenon was also predicted by Landau,[8] and was called by him the "zeroth sound". Because only the relation between ω and τ is essential in fixing the nature of the sound, these two kinds of sound can be described as low-frequency sound ($\omega\tau \ll 1$) and high-frequency sound ($\omega\tau \gg 1$).

The speed of sound at low frequencies or not too low temperatures, for which the condition $\omega\tau \ll 1$ holds, is determined by the compressibility in the usual way. It was shown by Landau[1] that there is an important dependence of the speed on the function f.

It is convenient to express the compressibility in terms of the derivative $\partial\mu/\mu N$. Here N is the total number of particles. Using the fact that the chemical potential μ depends only on N/V, we find:

$$\frac{\partial\mu}{\partial N} = -\frac{V\frac{\partial\mu}{\partial V}}{N} = -\frac{V^2}{N^2}\frac{\partial p}{\partial V}. \tag{8.1}$$

As is well known, the speed of sound is given by the relation

$$c^2 = \frac{\partial p}{\partial \rho} = \frac{\partial p}{\partial\left(\frac{mN}{V}\right)} = \frac{1}{m}\left(N\frac{\partial\mu}{\partial N}\right). \tag{8.2}$$

The derivative $\partial\mu/\partial N$ is calculated in the following way. Because $\mu = \epsilon(p_0)$, the change of μ occurs both on account of the change of p_0 and also because of the change of the form of the function $\epsilon(p)$:

$$\delta\mu = \int f\,\delta n'\,d\tau' + \frac{\partial\varepsilon_0}{\partial p_0}\delta p_0. \tag{8.3}$$

The changes δN and δp are connected by the relation

$$\delta N = 8\pi p_0^2\,\delta p_0\frac{V}{(2\pi\hbar)^3}. \tag{8.4}$$

Since only changes δn close to the Fermi limit are important in the integral of Eq. (8.3), the integration over the absolute value of the momentum can be performed. This gives

$$\int f\,\delta n'\,d\tau' = \frac{\delta N}{4\pi V}\int f\,d\Omega. \tag{8.5}$$

Substitution in Eq. (8.3) gives [cf. also Eqs. (2.1) and (2.2)]:

$$\frac{\partial\mu}{\partial N} = \frac{1}{4\pi V}\int f\,d\Omega + \frac{(2\pi\hbar)^3}{8\pi p_0 m^* V}. \tag{8.6}$$

Equation (2.5) enables us to express the effective mass m* appearing here in terms of the actual mass m of the atoms. Substituting the resulting expression in Eq. (8.2), and using the fact that

$$N=\frac{8\pi p_0^3 V}{3(2\pi h)^3},$$

we find:

$$c^2=\frac{p_0^2}{3m^2}+\frac{2}{3m}\left(\frac{p_0}{2\pi h}\right)^3\int f(\chi)(1-\cos\chi)\,d\Omega. \quad (8.7)$$

The formulas (2.5) and (8.7) make it possible to get some information about the function f(χ). If we introduce the notation

$$F(\chi)=f(\chi)\left(\frac{d\tau}{d\varepsilon}\right)_{\varepsilon=\mu}=f(\chi)\frac{p_0 m^*}{\pi^2 h^3}=\sum_k F_k P_k(\cos\chi), \quad (8.8)$$

the first coefficients of the expansion in Legendre polynomials can be found from Eqs. (2.5) and (8.7). Indeed, it is not hard to get the relations

$$\frac{1}{m}=\frac{1}{m^*}(1+\overline{F\cos\chi}),\quad c^2=\left(\frac{p_0^2}{3m^2}\right)\frac{1+\overline{F}}{1+\overline{F\cos\chi}}. \quad (8.9)$$

Substituting the values of the parameters for He^3 from Eq. (3.4) and the value of the speed of sound,[9] $c_{T=0}=183$ m/sec, we find: d:

$$F_0=\overline{F}=4.65,\quad F_1=3\overline{F\cos\chi}=1.3. \quad (8.10)$$

In view of the fact that the later coefficients cannot be determined in such a simple way, hereafter we shall confine ourselves in specific cases to the first two harmonic contributions to the function F:

$$F=F_0+F_1\cos\chi. \quad (8.11)$$

We now go on to the study of the "zeroth sound", i.e., the case in which $\omega\tau\gg 1$. The kinetic equation has the form (5.4). We substitute in it the distribution function in the form

$$n=n_0+\delta n, \quad (8.12)$$

where the small deviation $\delta n\sim e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. Here, of course, we have to take into account the fact that, according to Eq. (1.5), the energy of the excitations also undergoes a change. Inasmuch as the collision integral has the behavior $I(n)\sim\delta n/\tau$, and $\omega\tau\gg 1$, we can neglect I(n).

Thus we find that

$$(\mathbf{k}\cdot\mathbf{v}-\omega)\,\delta n-\mathbf{k}\cdot\mathbf{v}\frac{\partial n_0}{\partial\varepsilon}\int f\,\delta n'\,\delta\tau'=0.$$

From the form of this equation it follows that δn is proportional to $\partial n_0/\partial\epsilon$. Writing $\delta n=\partial n_0/\partial\epsilon\,\nu$, we get:

$$(\mathbf{k}\cdot\mathbf{v}-\omega)\nu+\mathbf{k}\cdot\mathbf{v}\int F\nu'\frac{d\Omega'}{4\pi}=0. \quad (8.13)$$

If we take **k** as the polar axis and introduce the symbols $u=\omega/k$ for the speed of propagation of the wave and $s=u/v$, then Eq. (8.13) takes the form:

$$(s-\cos\theta)\,\nu(\theta,\varphi)=\cos\theta\int F(\chi)\,\nu(\theta',\varphi')\frac{d\Omega}{4\pi}. \quad (8.14)$$

From this we can determine $\nu(\theta,\varphi)$ and the characteristic value s. In view of the fact that the change of the distribution function, δn, is proportional to $\partial n_0/\partial\epsilon$, it is clear that the essential point is the deformation of the Fermi surface. The shape of this deformation is determined by the function ν. Equation (8.14) depends essentially on the function F.

Let us consider first as an example the simplest case, namely $F=F_0=$ const. With this we get from Eq. (8.14) (without the periodic exponential factor)

$$\nu=\frac{\text{const}\,t\cdot\cos\theta}{s-\cos\theta}. \quad (8.15)$$

A displacement of the Fermi surface as a whole (this corresponds to ordinary sound) would be given by $\nu\sim\cos\theta$. Thus in the present case we have to do with an anisotropic deformation of the Fermi surface. As we shall see latter, s must be larger than unity. Thus the Fermi surface is stretched out in the direction of the motion.

Substituting Eq. (8.15) in Eq. (8.14) with $F=F_0$, we find the equation for s. Carrying out the elementary integration, we find:

$$\frac{s}{2}\ln\frac{s+1}{s-1}-1=\frac{1}{F_0}. \quad (8.16)$$

From this it can be seen that if s is real (which corresponds to undamped waves) it must be larger than 1, i.e.,

$$u>v. \quad (8.17)$$

From Eq. (8.14) it can be seen that this condition remains valid for any function F. Furthermore, since the left member of Eq. (8.16) is always positive, it is clear that the condition for the existence of zeroth sound is $F_0>0$.

If the function F_0 is large, s is also large. From Eq. (8.16) we get $s\to(F_0/3)^{1/2}$ for $F_0\to\infty$. On the other hand, for $F_0\to 0$ we have $s\to 1$, i.e., $u\to v$. This is the case of an almost free Fermi gas.

It is not hard to see that the conclusion that $s\to 1$ for $F\to 0$ does not depend on the form of F. In fact, it follows from Eq. (8.14) that for $F\to 0$, $s\to 1$, and ν is different from zero only for small θ. According to Eq. (8.7), in a weakly nonideal Fermi gas $c^2\approx p_0^2/3m$, i.e., $c\cong 3^{-1/2}u$. Thus the speed of the zeroth sound will exceed the ordinary speed of sound by a factor $3^{1/2}$.

Let us now consider the general case F(χ).[10] We represent this function as a sum of spherical harmonics as in Eq. (8.8). We substitute Eq. (8.8) in Eq. (8.14) and use the addition theorem of Legendre polynomials

$$P_n(\chi) = \sum_{m=-n}^{n} P_n^m(\theta)\, P_n^m(\theta')\, e^{im(\varphi-\varphi')} \frac{(n-|m|)!}{(n+|m|)!},$$

where $P_n^m = P_n^{-m}$ are the associated Legendre polynomials. After the indicated substitution we get:

$$(\cos\theta - s)\nu + \cos\theta \sum \frac{(n-|m|)!}{(n+|m|)!} P_n^m(\theta) F_n e^{im\varphi} \times \int P_n^m(\theta')\, \nu(\theta', \varphi')\, e^{-im\varphi'} \frac{d\Omega'}{4\pi} = 0. \tag{8.18}$$

We introduce the notation

$$F_n \frac{(n-|m|)!}{(n+|m|)!} \int P_n^m(\theta')\, \nu(\theta'\varphi')\, e^{-im\varphi'} \frac{d\Omega'}{4\pi} = \Phi_{nm} \tag{8.19}$$

and solve Eq. (8.18) for ν:

$$\nu = \frac{\cos\theta}{\cos\theta - s} \sum \Phi_{nm} P_n^m(\theta)\, e^{im\varphi}. \tag{8.20}$$

Substituting this expression into Eq. (8.19) and carrying out the integration over φ we get:

$$F_n \frac{(n-|m|)!}{(n+|m|)!} \int \sum_k P_n^m(\theta') \frac{\cos\theta'}{\cos\theta' - s} \cdot P_k^m(\theta') \frac{d\Omega'}{4\pi} \Phi_{km} = \sum_k \Phi_{km}\delta_{kn}. \tag{8.21}$$

Thus we have obtained a system of homogeneous equations for the quantities Φ_{km}. This system separates into independent subsystems corresponding to different values of m. From Eq. (8.21) it follows that in the Fermi fluid at absolute zero vibrations of several different types can be propagated, characterized by different dependences of the amplitude on the angles θ, φ. The value m = 0 corresponds to vibrations in which ν is isotropic in the plane perpendicular to **k**. For m ≠ 0 the vibrations are polarized in a definite way in this plane. The number of types of vibrations is determined by the number of possible values of m ($|m| \leq n$). The speeds of propagation of the vibrations are determined by the requirement that the determinants of the corresponding subsystems of equations be equal to zero:

$$\begin{gathered} \| \delta_{kn} + F_n \Omega_{kn}^m(s) \| = 0 \quad (N \geqslant n,\ k \geqslant |m|) \\ \Omega_{kn}^m(s) = \frac{(n-|m|)!}{(n+|m|)!} \int P_k^m(\theta') \frac{\cos(\theta')}{\cos\theta' - s} P_n^m(\theta') \frac{d\Omega'}{4\pi}. \end{gathered} \tag{8.22}$$

Because $P_n^{-m} = P_n^m$, the coefficients Ω_{kn}^m do not depend on the sign of m, so that vibrations that differ only by the sign of m are propagated with the same speed.

From Eq. (8.22) it can be seen that the equations for the speeds are transcendental equations. In the general case they do not always have real roots. Cases are also possible, however, in which there are several real roots. Then there are several types of vibrations with the same polarization in the plane perpendicular to **k**.

As an example let us consider the case in which the function F(χ) contains only the zeroth and first harmonics [the function (8.11)]. Here the coefficients Ω_{kn}^m are given by

$$\Omega_{00}^0 = \frac{1}{2}\int_{-1}^{1} \frac{x\,dx}{x-s} = 1 - \frac{s}{2}\ln\frac{s+1}{s-1} = -w,$$

$$\Omega_{10}^0 = \Omega_{01}^0 = \frac{1}{2}\int_{-1}^{1} \frac{x^2 dx}{x-s} = -sw,$$

$$\Omega_{11}^0 = \frac{1}{2}\int_{-1}^{1} \frac{x^3\,dx}{x-s} = \frac{1}{3} - s^2 w,$$

$$\Omega_{11}^1 = \frac{1}{4}\int_{-1}^{1} \frac{(1-x^2)}{x-s}\,dx = \frac{1}{2}\left[(s^2-1)w - \frac{1}{3}\right].$$

On substituting in the determinant (8.22) we get for the speed of propagation of vibrations of type m = 0 the equation

$$w = \frac{1 + F_1/3}{F_0 + F_0F_1/3 + F_1 s^2}. \tag{8.23}$$

For the case m = 1 we get the equation

$$w = \frac{F_1 - 6}{3F_1(s^2-1)}. \tag{8.24}$$

This equation has one real root for $F_i > 6$.

Let us now look at the application of our formulas to the case of liquid He^3. According to Eq. (8.10) there is one speed for the vibrations of type m = 0 [the root of Eq. (8.23)]

$$s = \frac{u}{v} = 1.72, \quad u = 192 \text{ m/sec}.$$

The vibrations of type m = 1 (and also all with m > 1) are absent. It is of course possible that this conclusion is a consequence of the crudeness of the approximation we have taken for F(χ), but we see no reasons for thinking so.

In addition to the vibrations that have been considered, waves of an entirely different type can be propagated in a Fermi fluid at absolute zero; these can be called spin waves.[8] In the treatment of these vibrations one must take into account the spin part of the function f, which is given by Eq. (4.3). We introduce a new symbol

$$\Phi(\chi) = f(\chi)\left(\frac{d\tau}{d\varepsilon}\right)_{\varepsilon=\mu} = F(\chi) + Z(\chi)\,\boldsymbol{\sigma}\cdot\boldsymbol{\sigma}'. \tag{8.25}$$

The equation for zeroth sound now takes the form

$$(s-\cos\theta)\,\nu=\cos\theta\cdot\frac{1}{2}\,\mathrm{Sp}_{\sigma'}\frac{1}{4\pi}\int\Phi\nu'\,d\Omega'. \qquad (8.26)$$

If we assume that ν does not depend on the spin, this goes over into Eq. (8.14). It is also possible, however, to assume a different form for the function ν, namely to take

$$\nu=\boldsymbol{\nu}\cdot\boldsymbol{\sigma} \qquad (8.27)$$

For this function we get the equation

$$(s-\cos\theta)\,\boldsymbol{\nu}=\cos\theta\int z\boldsymbol{\nu}'\frac{d\Omega'}{16\pi}. \qquad (8.28)$$

Thus we have for the components of the vector $\boldsymbol{\nu}$ an equation that differs from Eq. (8.14) only by the replacement of F by Z/4. Therefore all the further developments must be the same as those carried out above.

According to the results of Sec. 4, for the case of He^3 the quantity Z_0 is approximately -3.4, that is, a negative quantity. Although it is of course impossible to draw any rigorous conclusions from this, still it is most probable that no spin waves can be propagated in He^3.

The possibility of the propagation of sound waves at T = 0 means that the Fermi fluid has a Bose branch of the energy spectrum, in the form of phonons with energy ϵ = up. But the contributions from this branch to the thermodynamic quantities involve higher powers of T (heat capacity ~ T^3), which were not included in the approximate theory that has been considered.

9. THE DISPERSION AND ABSORPTION OF SOUND

In the preceding section we considered the two limiting cases $\omega\tau \ll 1$ and $\omega\tau \gg 1$. Let us now examine how the transition occurs from ordinary sound to zeroth sound, i.e., let us consider the dispersion of sound.[10] Because the region of the dispersion, $\omega\tau \sim 1$, is at the same time the region of strong absorption, our treatment will also enable us to deal with the problem of the absorption of sound.

In the present case we have to find the solution of the kinetic equation, including the collision integral. Because the interaction law of the excitations is still unknown, however, we shall simplify the form of the collision integral.

For this purpose we could introduce a certain effective time τ and replace the collision integral I in Eq. (5.1) by the expression $-\delta n/\tau$. With this replacement, however, the kinetic equation will not yield any conservation laws for the number of excitations, the momentum, and the energy, and this makes the transition to hydrodynamics impossible. Since both the second (dilatational) viscosity and the effects of thermal conductivity (see below) are negligibly small in our case, the collision integral contains no terms in the zeroth and first spherical harmonics. We therefore replace the collision integral by the following expression:*

$$I(n)\longrightarrow -\tau^{-1}\left(\delta n-\overline{\delta n}-\overline{3\,\delta n\cos\theta}\,\cos\theta\right). \qquad (9.1)$$

It is easy to see that the integration over $d\tau_p$ makes this expression go to zero. It also goes to zero after multiplication by ϵ or by p cos θ and integration over $d\tau_p$.† Thus the equations for the conservation of the number of particles, the momentum, and the energy are automatically satisfied. According to Eqs. (8.14) and (9.1) the basic kinetic equation now takes the following form:

$$\begin{aligned}(kv\cos\theta-\omega)\,\nu+kv\cos\theta\int F\nu'\frac{d\Omega'}{4\pi}\\ =-\frac{1}{i\tau}\left(\nu-\overline{\nu}-\overline{3\nu\cos\theta}\cdot\cos\theta\right).\end{aligned} \qquad (9.2)$$

Having in mind the application of the results to He^3, in order not to complicate the problem we shall write the function F in the two-term form of Eq. (8.11).

We introduce the notation

$$\overline{\nu}=\nu_0;\ \overline{3\nu\cos\theta}=\nu_1;\ \sigma=-i\tau kv;\ \xi=\frac{i\omega\tau-1}{i\tau kv}, \qquad (9.3)$$

after which we get without difficulty from Eq. (9.2):

$$(\cos\theta-\xi)\,\nu+\cos\theta\left(F_0\nu_0+\frac{1}{3}F_1\nu_1\cos\theta\right)=\frac{1}{\sigma}(\nu_0+\nu_1\cos\theta). \qquad (9.4)$$

We now solve the equation for ν and calculate $\overline{\nu} = \nu_0$ and $\overline{\nu\cos\theta} = \nu_1/3$. We thus find two equations for the two quantities ν_0 and ν_1:

$$\nu_0=F_0\nu_0w+\frac{1}{3}\,wF_1\nu_1-\frac{1}{\sigma}\,\frac{w+1}{\xi}\,\nu_0-\frac{\nu_1}{\sigma}\,w, \qquad (9.5)$$

$$\frac{1}{3}\nu_1=F_0\nu_0\xi w-\frac{1}{3}\left(\frac{1}{3}-\xi^2w\right)F_1\nu_1-\frac{w}{\sigma}\nu_0-\frac{\xi w}{\sigma}\nu_1, \qquad (9.6)$$

where $w=\frac{1}{2}\ln\frac{\xi+1}{\xi-1}-1$.

From the condition for compatibility of (9.5) and (9.6) we get an equation for the complex speed of sound

$$\begin{aligned}\left(1+\frac{1}{\xi\sigma}\right)\left(1+\frac{F_1}{3}\right)-w\left\{\left(1+\frac{F_1}{3}\right)\left(F_0-\frac{1}{\xi\sigma}\right)\right.\\ \left.+\xi^2\left(F_1-\frac{3}{\xi\sigma}\right)\cdot\left(1+\frac{1}{\xi\sigma}\right)\right\}=0.\end{aligned} \qquad (9.7)$$

This is the desired equation for the frequency

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*Here and in what follows the bar denotes averaging over the angles.

†In this integration only the range of values of momentum close to p_0 is of importance, since the function δn contains a δ-function singularity for $\varepsilon = \mu$.

dependence of the speed of sound or, in other words, for the dispersion of sound in the Fermi fluid. We first consider two extreme cases.

a) Low frequencies, $\omega\tau \ll 1$. Here we have $\sigma \to 0$, $\xi\sigma \to -1$, and $\xi \to \infty$. Expansion of w in powers of $1/\xi$ gives

$$w=\frac{1}{3\xi^2}+\frac{1}{5\xi^4}$$

and after simple manipulations Eq. (9.7) takes the form

$$\left(1+\frac{1}{\xi\sigma}\right)^2=\frac{1}{3\xi^2}\cdot\left(1+\frac{F_1}{3}\right)\cdot\left(F_0-\frac{1}{\xi\sigma}\right)$$
$$+\frac{1}{5\xi^2}\left(1+\frac{1}{\xi\sigma}\right)\cdot\left(F_1-\frac{3}{\xi\sigma}\right). \qquad (9.8)$$

Recalling the relations (9.3), we have:

$$\left(\frac{1+\xi\sigma}{\sigma}\right)^2=\left(\frac{\omega}{kv}\right)^2.$$

From Eqs. (9.8) and (9.3) we find in first order in $i\omega\tau$

$$\left(\frac{\omega}{kv}\right)^2=\frac{1}{3}(1+F_0)\left(1+\frac{F_1}{3}\right)-\frac{4}{15}i\omega\tau\left(1+\frac{F_1}{3}\right). \qquad (9.9)$$

The first term corresponds to the speed of ordinary sound in the Fermi fluid. The absorption of sound in the region $\omega\tau \ll 1$ is found by the elementary rule from Eq. (9.9) as the imaginary part of the wave vector:

$$\gamma=\operatorname{Im}k=\frac{2\omega^2\tau v^2}{15c^3}\left(1+\frac{F_1}{3}\right). \qquad (9.10)$$

In view of the fact that the present case corresponds to ordinary sound, we can use instead of Eq. (9.10) the more precise ordinary formula

$$\gamma=\frac{\omega^2}{2\rho c^3}\left\{\left(\frac{4}{3}\eta+\zeta\right)+\frac{\varkappa}{c_p}\left(\frac{c_p}{c_v}-1\right)\right\}.$$

Noting that $\zeta \ll \eta$ and also that κ/c_p, which is a quantity of the order of η, is multiplied by the small factor $(c_p/c_V)-1$ [it is not hard to verify that it is of the order $(T/\mu)^2$], we find:

$$\gamma=\frac{2\omega^2}{3\rho c^3}\eta. \qquad (9.11)$$

Comparing this expression with Eq. (9.10), we can determine the time τ from experimental data:

$$\eta=\frac{1}{5}\rho\tau v^2\left(1+\frac{1}{3}F_1\right). \qquad (9.12)$$

From the data of K. N. Zinov'eva on the viscosity[7] we get

$$\tau\cong 2.3\cdot10^{-12}\cdot T^{-2}\ \text{sec}\ (T\ \text{in}\ {}^\circ\text{K}) \qquad (9.13)$$

The corresponding value for the absorption coefficient is

$$\gamma\sim 1\cdot10^{-17}\left(\frac{\omega}{T}\right)^2\text{cm}^{-1}. \qquad (9.14)$$

b) We now consider the second extreme case, that of high frequencies and low temperatures, $\omega\tau \gg 1$. In this case

$$\sigma\to\infty;\ \xi\sigma\to\infty;\ \xi=s+i\xi',\ |\xi'|\ll s.$$

Equation (9.7) takes the form

$$\left(1+\frac{F_1}{3}\right)-w(s)\left\{\left(1+\frac{F_1}{3}\right)F_0+s^2F_1\right\}=0. \qquad (9.15)$$

This equation agrees exactly with the equation (8.23) that determines the speed of the zeroth sound.

As for the absorption of the zeroth sound, to calculate it we must find the imaginary part ξ' of the speed of sound. From Eq. (9.9) we get the equation

$$\xi'\left\{\frac{1}{w(s)}\left(1+\frac{F_1}{3}\right)\left(\frac{s}{s^2-1}-\frac{w(s)+1}{s}\right)-2sw(s)F_1\right\}$$
$$-\frac{1}{\omega\tau}\left\{\left(1+\frac{F_1}{3}\right)(1+w(s))+w(s)s^2(3-F_1)\right\}=0. \qquad (9.16)$$

By means of Eq. (9.3) we find the absorption coefficient

$$\gamma=\operatorname{Im}k=\frac{1}{s\tau v}-\frac{\omega\xi'}{s^2v}. \qquad (9.17)$$

If we substitute here the values of the parameters for He^3 ($s = 1.84$, $v = 1.13\times10^4$ cm/sec), we get:

$$\xi'=\frac{1.5}{\omega\tau},\quad \gamma\sim3\cdot10^6\cdot T^2\text{cm}^{-1}. \qquad (9.18)$$

Thus the absorption of the zeroth sound does not depend on the frequency and increases with increasing temperature as $1/\tau$, i.e., proportional to T^2.

In this calculation the speed of sound has been treated classically. The validity of this treatment depends on the inequality $\hbar\omega \ll T$. For the case $\hbar\omega \gtrsim T$ it is necessary to treat the problem by quantum theory. The decrease of the number of sound quanta per unit time, due to the collisions of the Fermi particles, is given by

$$\int w\{n_1n_2(1-n_1')(1-n_2')-n_1'n_2'(1-n_1)(1-n_2)\}$$
$$\times\delta(\mathbf{p}'+\mathbf{p}_2'-\mathbf{p}_1-\mathbf{p}_2-\hbar\mathbf{k}) \qquad (9.19)$$
$$\times\delta(\varepsilon_1'+\varepsilon_2'-\varepsilon_1-\varepsilon_2-\hbar\omega)\,d\tau_1\,d\tau_2\,d\tau_1'\,d\mathbf{p}_2'.$$

The function w, which depends on the four momenta, is unknown. It is possible, however, to carry out the calculation in such a way that the absorption coefficient is expressed in terms of its classical value (9.17); this has been done in a paper by Landau.[8]

Making use of the fact that in the region in which the Fermi distribution falls off (which is the important one for the problem) the only rapidly varying function is the occupation number n(ϵ), we can assume that the angular integrals give a certain constant, which does not change as we go from the classical region $\hbar\omega \ll T$ to the quantum region $\hbar\omega \gg T$. There then remains only the integral over the energies

$$I = \int \{n_1 n_2 (1-n_1')(1-n_2') - n_1' n_2' (1-n_1)(1-n_2) \times \delta(\varepsilon_1' + \varepsilon_2' - \varepsilon_1 - \varepsilon_2 - \hbar\omega)\, d\varepsilon_1\, d\varepsilon_2\, d\varepsilon_1'\, d\varepsilon_2'. \tag{9.20}$$

Substituting the Fermi function for n and extending the integration from $-\infty$ to ∞ (this is permissible because of the rapid convergence of the integrals), after a rather lengthy calculation we get

$$I = T^3 \frac{2\pi^2}{3} \hbar\omega \left[1 + \left(\frac{\hbar\omega}{2\pi T}\right)^2\right]. \tag{9.21}$$

The absorption coefficient is proportional to the integral I. From the classical limiting case we can determine the proportionality factor. Since the classical value of γ must be obtained for $\hbar\omega \ll T$, we have in the general case

$$\gamma = \gamma_{\text{cl}} \left[1 + \left(\frac{\hbar\omega}{2\pi T}\right)^2\right], \tag{9.22}$$

where γ_{cl} is given by Eq. (9.17). In view of the fact that γ_{cl} is proportional to T^2, in the quantum limiting case $\hbar\omega \gg T$ the absorption coefficient γ must be proportional to ω^2.

10. THE SCATTERING OF LIGHT (FLUCTUATIONS OF THE DISTRIBUTION FUNCTION)

According to the foregoing, a particular condition for the possibility of the propagation of "zeroth sound" is the inequality

$$\omega_{\text{sound}}\tau \gg 1, \tag{10.1}$$

where τ is the time between collisions. For He^3 this is of the order of $10^{-12}T^{-2}$ sec. If we take even a temperature ~ 0.01°K, a frequency higher than 10^8 cps is required for the direct observation of zeroth sound, and this makes the performance of such an experiment very difficult.

Besides the direct experiment, an indirect method can be proposed, which consists of the observation of the Rayleigh scattering of light in liquid He^3.* As is well known, in Rayleigh scattering there appear, in addition to the main line, satellites that differ from it in frequency by

$$\Delta\omega = \pm 2 \frac{u}{c} \omega \sin \frac{\theta}{2},$$

where u is the speed of sound and θ is the scattering angle. The speed of the zeroth sound in He^3 is of the order of 2×10^4 cm/sec, that is, $\Delta\omega \sim 10^{-6}\omega$. Thus the observation of the frequency distribution of the scattered light provides in principle a possibility for measuring the speed of the zeroth sound. The condition (10.1) can be satisfied owing to the high frequency of visible light.

In addition to this side of the matter, the scattering of light in a Fermi fluid at sufficiently low temperatures has a number of specific features which give interest to the theoretical study of this phenomenon, in particular the frequency distribution of the intensity.*

As is well known, the frequency and angular distributions for the Rayleigh scattering of unpolarized light are given by the formula[12]†

$$dh = \frac{\omega^4}{6\pi c^4} \cdot \frac{1}{2\pi V} \overline{\left| \int \delta D_{\Delta\omega}(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}} dV \right|^2} \frac{3}{4} (1 + \cos^2\theta) \frac{d\Omega}{4\pi} d\Delta\omega, \tag{10.2}$$

where ω is the frequency of the incident light, θ is the scattering angle, q is the change of the wave vector of the light, of magnitude $(2\omega/c) \sin(\theta/2)$, and $\delta D_{\Delta\omega}$ is the Fourier component of the fluctuation $\delta D(t)$ of the dielectric constant:

$$\delta D_{\Delta\omega} = \frac{1}{\sqrt{t_0}} \int_0^{t_0} \delta D(t)\, e^{i\Delta\omega t} d\tau, \tag{10.3}$$

where t_0 is a certain large value, which will be let go to infinity in the final formula.

The bar in Eq. (10.2) means averaging over the fluctuations. Hereafter we shall for simplicity take the volume of the system to be unity.

Because of the very small polarizability of helium atoms we can assume that the change of the dielectric constant occurs as a result of density fluctuations, i.e., $\delta D = (\partial D/\partial N)\delta N$, where N is the number of particles in unit volume. But according to the general theory of the Fermi fluid the number of excitations is equal to the number of atoms in the fluid. Thus we can write

$$\int \delta D_{\Delta\omega}(\mathbf{r})\, e^{-i\mathbf{q}\cdot\mathbf{r}} dV = \frac{\partial D}{\partial N} \int \delta n_{\mathbf{q},\,\Delta\omega}(\mathbf{p})\, d\tau_p, \tag{10.4}$$

where $\delta n_{\mathbf{q},\,\Delta\omega}(\mathbf{p})$ is the Fourier component with respect to r and t [the latter in the sense of Eq. (10.3)] of the fluctuation of the distribution function of the excitations.

Before going on to further calculations, let us

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*The idea of using the Rayleigh scattering was first proposed by S. P. Kapitza.

*We note that in the case of high temperatures, for which $\omega_{\text{sound}} \ll 1$, the scattering of light will be described by the usual formulas (cf. reference 12).

†Here dh is the so-called differential extinction coefficient. The integral of dh with respect to dΩ and dΔω gives the total extinction coefficient h, which is the damping decrement of the photon flux density in the medium.

note one important fact. In Eq. (10.2) the averaging is taken over all possible fluctuations. In the range of temperatures and frequencies in which $\hbar\Delta\omega \gtrsim T$ we must take account of quantum effects in carrying out this averaging. For this, however, (cf. reference 12) it is enough to know the result for the purely classical case, that is, for $T \gg \hbar\Delta\omega$, and then introduce a certain correction factor. For scattering with increase of the frequency by the amount $\Delta\omega$ (anti-Stokes scattering) we must introduce the factor $(h\Delta\omega/T)N(\Delta\omega)$, and for scattering with decrease of the frequency by $\Delta\omega$ (Stokes scattering) the factor is $(h\Delta\omega/T)[N(\Delta\omega) + 1]$, where $N(\Delta\omega)$ is the Bose distribution function. If we use negative values of $\Delta\omega$ for the description of the Stokes scattering, then owing to the relation $N(-\Delta\omega) + 1 = -N(\Delta\omega)$, it turns out that the correction factor for both cases has the form

$$\frac{\hbar\Delta\omega}{T}\left[e^{\frac{\hbar\Delta\omega}{T}} - 1\right]^{-1}. \tag{10.5}$$

Thus we shall take $T \gg \hbar\omega$. To find the fluctuations of the distribution function we shall use the method proposed by Rytov[13] and by Landau and Lifshitz[14] for calculating fluctuations in electrodynamics and hydrodynamics.* By this method we find the fluctuation of the "random force" appearing in the kinetic equation; then by solving this equation we can also get the fluctuation of the distribution function.

For the case of the Fermi fluid we shall start with the kinetic equation, which we write in the form

$$\frac{\partial\delta n}{\partial t} + \frac{\partial\delta n}{\partial \mathbf{r}}\cdot\frac{\partial\varepsilon}{\partial \mathbf{p}} - \frac{\partial n_0}{\partial \mathbf{p}}\int f(\mathbf{p}, \mathbf{p}')\frac{\partial\delta n(\mathbf{p}')}{\partial \mathbf{r}}\,d\tau_{p'} = I(\delta n) + y(\mathbf{p}, \mathbf{r}, t). \tag{10.6}$$

Except for the "random force" $y(\mathbf{r}, \mathbf{p}, t)$ this equation is an approximation to the kinetic equation (5.1) that is linear in δn.

In what follows we shall be concerned only with the case of frequencies and temperatures satisfying the relation (10.1), i.e., with the case in which the collisions can be neglected. The detailed form of the collision integral is then of no importance, since it plays the part of an auxiliary quantity in the calculations and can be set equal to zero in the final result. In view of this we set

$$I(\delta n) = -\frac{\delta n}{\tau}, \tag{10.7}$$

where τ is a large quantity. We have also to find the rate of change of the entropy. Recalling that the number of particles and the total energy are prescribed, and using the relation

$$\delta\varepsilon(\mathbf{p}) = \int f(\mathbf{p}, \mathbf{p}')\,\delta n(\mathbf{p}')\,d\tau_{p'}, \tag{10.8}$$

we find:

$$S = -\left\{\int \frac{\delta n\,[I(\delta n) + y]}{n_0(1-n_0)}\,d\tau_p\,dV + \frac{1}{T}\int f(\mathbf{p}, \mathbf{p}')\,\delta(\mathbf{r}-\mathbf{r}')\,\delta n I(\delta n')\,d\tau_p\,dV\,d\tau_{p'}\,dV'\right\}. \tag{10.9}$$

Noting that $n_0(1 - n_0) \approx T\delta(\epsilon - \mu)$, where μ is the chemical potential, we readily see that $\delta n(\mathbf{p})$ must have the form

$$\delta n(\mathbf{p}) = \nu(\theta, \varphi)\cdot\delta(\varepsilon - \mu), \tag{10.10}$$

where θ and φ are the polar angles of the vector $\mathbf{p}$. This formula means that the fluctuations of the distribution function occur only in the neighborhood of the Fermi surface.

It is natural to take a similar form for $\mathbf{y}$, i.e.,

$$y(\mathbf{p}) = y^{\varepsilon}(\theta, \varphi)\,\delta(\varepsilon - \mu). \tag{10.11}$$

We now introduce the notation

$$F(\chi) = \left[f(\mathbf{p}, \mathbf{p}')\frac{d\tau}{d\varepsilon}\right]_{\varepsilon=\varepsilon'=\mu},$$

where χ is the angle between $\mathbf{p}$ and $\mathbf{p}'$, and expand δn, y, and F in spherical harmonics

$$\left.\begin{aligned}\nu(\theta, \varphi) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_n^m P_n^m(\cos\theta)\,e^{im\varphi},\\ y^{\varepsilon}(\theta, \varphi) &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} y_n^m P_n^m(\cos\theta)\,e^{im\varphi},\\ F(\chi) &= \sum_{n=0}^{\infty} F_n P_n(\cos\chi).\end{aligned}\right\} \tag{10.12}$$

Since ν and y^{ϵ} are real quantities, we have

$$A_n^m = (A_n^{-m})^*,\quad y_n^m = (y_n^{-m})^*.$$

Using the expression (10.7) for the collision integral, we get the following formula for the rate of change of the entropy:

$$\dot{S} = \int dV\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}\frac{1}{T}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{F_n}{2n+1} + 1\right) \times\frac{1}{2n+1}\left(\frac{(n+|m|)!}{(n-|m|)!}\right)\left(\frac{A_n^m}{\tau} - y_n^m\right)A_n^{-m}. \tag{10.13}$$

Let us now introduce the notation

$$\dot{x}_n^m = -\frac{A_n^m}{\tau} + y_n^m. \tag{10.14}$$

Then in order for the expression (10.13) to take the form (cf. reference 14)

$$\dot{S} = -\sum_i X_i\cdot\dot{x}_j$$

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*The writers express their gratitude to L. P. Gor'kov, I. E. Dzyaloshinskii, and L. P. Pitaevskii, who called attention to the possibility of applying this method to the kinetic equation.

we must take as the generalized forces X_i the expressions (10.15)

$$X_n^m = \frac{1}{T}\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}\left(\frac{F_n}{2n+1}+1\right)\frac{1}{2n+1}\cdot\frac{(n+|m|)!}{(n-|m|)!}A_n^{-m}.$$

In the expression (10.14) the quantity y_n^m plays the role of the "random force." Writing this relation in the form

$$x_n^m = -\sum_{n'm'} \gamma_{n,n'}^{m,m'} X_{n'}^{m'} + y_n^m,$$

where the coefficients γ can be determined without difficulty from Eq. (10.15), we have according to the general theory of fluctuations:

$$\begin{aligned} y_n^m(\mathbf{r},t)\, y_{n'}^{m'}(\mathbf{r}',t') &= (\gamma_{n,n'}^{m,m'} + \gamma_{n',n}^{m',m})\,\delta(t-t')\,\delta(\mathbf{r}-\mathbf{r}') \\ &= \frac{2}{\tau}\,\delta_{n,n'}\delta_{m,-m'}\,\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \qquad (10.16) \\ &\times\left[T\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}\left(\frac{F_n}{2n+1}+1\right)\frac{1}{2n+1}\frac{(n+|m|)!}{(n-|m|)!}\right]^{-1}. \end{aligned}$$

Finally, using Eq. (10.12) and the relation

$$\sum_n (2n+1)P_n(\cos\chi) = 2\delta(\cos\chi - 1),$$

we get after some transformations the general relation

$$\overline{y(\mathbf{p},\mathbf{r},t)\,y(\mathbf{p}',\mathbf{r}',t')} = \frac{2T}{\tau}\delta(\mathbf{r}-\mathbf{r}')\,\delta(t-t') \qquad (10.17)$$

$$\times\left\{\frac{(2\pi\hbar)^3}{2}\delta(\mathbf{p}-\mathbf{p}')\,\delta(\varepsilon-\mu) + \delta(\varepsilon-\mu)\,\delta(\varepsilon'-\mu)\left(\frac{d\varepsilon}{d\tau_p}\right)_{\varepsilon=\mu}\right.$$

$$\left.\times\sum_{n=0}^{\infty}\frac{F_nP_n(\cos\chi)}{1+F_n/2n+1}\right\}.$$

By means of this formula and the kinetic equation (10.16) we can calculate the fluctuations of the distribution function, with which we are concerned here. Since in the general case of an arbitrary function f this is a rather complicated task, we confine ourselves to the case f = const.

Making use of the fact that the fluctuations occur only at the Fermi surface, and referring to Eq. (10.6), we express $\nu_{\mathbf{q},\Delta\omega}(\theta,\varphi)$ in terms of the corresponding Fourier component of $y^\epsilon(\theta,\varphi)$. This gives

$$\int \nu_{\mathbf{q},\Delta\omega}(\theta,\varphi)\frac{d\Omega}{4\pi} = \int\frac{y^{\varepsilon}_{\mathbf{q},\Delta\omega}(\theta,\varphi)\frac{d\Omega}{4\pi}}{-i\omega+\frac{1}{\tau}+i\mathbf{q}\mathbf{v}}\Bigg/\left(1+F_0\int\frac{i(\mathbf{q}\mathbf{v})\frac{d\Omega}{4\pi}}{-i\omega+\frac{1}{\tau}+i\mathbf{q}\mathbf{v}}\right), \qquad (10.18)$$

where $\mathbf{v} = (\partial\epsilon/\partial\mathbf{p})_{\epsilon=\mu}$. Averaging the square of the magnitude of this expression by means of Eq. (10.17)[here we have to remember that the Fourier components with respect to t are taken according to Eq. (10.3)], we find:

$$\overline{\left|\int\nu_{\mathbf{q},\Delta\omega}(\theta,\varphi)\frac{d\Omega}{4\pi}\right|^2} = 2\left(\frac{d\varepsilon}{d\tau_p}\right)_{\varepsilon=\mu}\frac{T}{\tau}\left[\frac{1}{2}\int_{-1}^{1}\frac{dx}{\left|qvx-\omega+\frac{i}{\tau}\right|^2}\right. \qquad (10.19)$$

$$\left.-\frac{F_0}{1+F_0}\left|\frac{1}{2}\int_{-1}\frac{dx}{qvx-\omega+\frac{i}{\tau}}\right|^2\right]\Bigg/\left|1+\frac{F_0}{2}\int_{-1}^{+1}\frac{qvx\,dx}{qvx-\omega+\frac{i}{\tau}}\right|^2.$$

We are interested in the limit of the value of this expression for $\tau\to\infty$. In the case in which $qv > |\Delta\omega|$, the denominator does not have any poles, and the important quantity is the residue in the integral in the numerator. We get as the result:

$$\frac{1}{2\pi}\overline{|\nu_{\mathbf{q},\Delta\omega}(\mathbf{p})\,d\tau_p|^2} = T\left(\frac{d\tau_p}{d\varepsilon}\right)\frac{1}{qv}\left\{\left[1+F_0\left(1-\frac{\Delta\omega}{2qv}\ln\frac{qv+\Delta\omega}{qv-\Delta\omega}\right)\right]^2+\left(\frac{F_0\Delta\omega\pi}{2qv}\right)^2\right\}^{-1}. \qquad (10.20)$$

In the opposite case, i.e., for $qv \ll \Delta\omega$, the pole in the denominator of the expression (10.19) is important. It is not hard to see that such a pole occurs, if $F_0 > 0$ and is given by $\Delta\omega = \pm sqv$, where s satisfies the equation

$$1+F_0\left[1-\frac{1}{2}\ln\left(\frac{s+1}{s-1}\right)\right]=0. \qquad (10.21)$$

This equation is identical with Eq. (8.16), which determines the speed of the zeroth sound. Using the relation

$$\frac{1}{\pi}\lim_{\tau\to\infty}\frac{\tau}{(\omega-\omega_0)^2+\tau^{-2}} = \delta(\omega-\omega_0),$$

we get without difficulty:

$$\frac{1}{2\pi}\overline{\left|\int\nu_{\mathbf{q},\Delta\omega}(\mathbf{p})\,d\tau_p\right|^2} = T\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}\frac{2(s^2-1)}{F_0(1+F_0-s^2)}$$

$$\times[\delta(\Delta\omega-sqv)+\delta(\Delta\omega+sqv)]. \qquad (10.22)$$

Thus the distribution of the scattered light in direction and frequency has the form [we have also introduced the quantum factor (10.5)]:

$$dh = \frac{\omega^4}{4\pi c^4}\left(\frac{\partial D}{\partial N}\right)^2\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu}(1+\cos^2\theta)\cdot\frac{\hbar\Delta\omega}{e^{\hbar\Delta\omega/T}-1}$$

$$\times\left[\frac{\theta(qv-|\Delta\omega|)}{2qv}\Bigg/\left\{\left[1+F_0\left(1-\frac{\Delta\omega}{2qv}\ln\frac{qv+\Delta\omega}{qv-\Delta\omega}\right)\right]^2\right.\right.$$

$$\left.+\left(\frac{F_0\Delta\omega\pi}{2qv}\right)^2\right\}+\frac{s^2-1}{F_0(1+F_0-s^2)}[\delta(\Delta\omega-sqv) \qquad (10.23)$$

$$\left.+\delta(\Delta\omega+sqv)]\right]\frac{d\Omega}{4\pi}\,d\Delta\omega,$$

where

$$\theta(y) = \begin{cases} 1 & y>0, \\ 0 & y<0. \end{cases}$$

This result has a simple physical meaning. As can easily be seen, the frequency spectrum consists of a central part $-qv < \Delta\omega < qv$ and two narrow lines at $\Delta\omega = \pm sqv$. The central part corresponds to the Doppler broadening of the main line. Comparison of Eq. (10.21) with Eq. (8.16) shows that the side lines are the satellites of the Rayleigh scattering, which appear because of the possibility of the propagation of zeroth sound ($sv = u$). The ratio between the intensities of the central part and the satellites in general depends on the scattering angle. In the two extreme cases, however, of high temperatures ($T \gg \hbar\omega u/c$) and low temperatures ($T \ll \hbar\omega u/c$) this ratio does not depend on the angle. One can carry out numerical computations for He^3, using unknown (sic) parameters. It then turns out that for the high-temperature case about 20 percent of the whole intensity is in the central part, and about 40 percent in each of the side lines. In the low-temperature case, owing to the quantum factor, the distribution is cut off on the side of positive $\Delta\omega$. In particular, the only one of the two satellites left is the Stokes line with $\Delta\omega = -uq$, which gets 90 percent of the intensity. In this case the central part receives only 10 percent of the whole intensity.

The total scattered intensity is obtained by integrating Eq. (10.23) over $d\Delta\omega$ and $d\Omega$. For the high-temperature case

$$\left(T \gg \hbar\Delta\omega \sim \hbar\omega\frac{u}{c}\right)$$

it is

$$h = \frac{\omega^4 T}{6\pi c^4}\left(\frac{\partial D}{\partial N}\right)^2\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu} J_1, \tag{10.24}$$

where J_1 is a numerical integral, which in the case of He^3 is about 0.5. In the quantum limiting case ($T \ll \hbar\omega u/c$) we have:

$$h = \frac{\hbar\omega^5 v}{6\pi c^4}\left(\frac{\partial D}{\partial N}\right)\left(\frac{d\tau_p}{d\varepsilon}\right)_{\varepsilon=\mu} J_2. \tag{10.25}$$

Here J_2 is a numerical integral, which in the case of He^3 is about 0.2. To carry out quantitative calculations from these expressions for the case of He^3, we need to know the value of $\partial D/\partial N_0$. There are no measurements of this quantity; therefore we take $D - 1$ to be proportional to N, and use the proportionality factor as given by the data on liquid He^4. For wavelength $\lambda = 5461$ A the index of refraction of liquid He^4 is 1.027. This gives $\partial D/\partial N = 2.5 \times 10^{-24}$. Substitution in Eqs. (10.24) and (10.25) gives (T in °K):

$$h(He^3)\, 10^{-69}\omega^4 T \text{ cm}^{-1} \quad \text{for} \quad \omega \ll 2\cdot 10^{17} T \text{ sec}^{-1},$$

$$h(He^3)\, 10^{-87}\omega^5 \text{ cm}^{-1} \quad \text{for} \quad \omega \gg 2\cdot 10^{17} T \text{ sec}^{-1}.$$

It must be remembered that the frequency must satisfy the relation (10.1) i.e., $\Delta\omega \gg 1/\tau$, or

$$\omega \gg 10^{18} T^2 \text{ sec}^{-1}.$$

If this condition is not fulfilled, the line width will be too large. Thus in the visible range of frequencies we need temperatures below 0.05°K. It is not hard to see that at temperatures of the order of 0.01° and frequencies in the visible region we have $h(He^3) \sim 10^{-9}\text{cm}^{-1}$, which is of course too small for the effect to be measurable.* But owing to the fact that ω occurs raised to a high power in the expression for h, it is possible that one may be able to measure the scattering in the ultraviolet region.

APPENDICES

MICROSCOPIC THEORY OF THE FERMI FLUID

A 1. The Rarefied Fermi Gas

In this section we consider the properties of a non-ideal Fermi gas in which the dimensions of the particles are small in comparison with the mean free path.[16] This model, a number of whose properties have been examined recently by Yang and Huang,[17] and also by Yang and Lee,[18] was called by those writers the hard-sphere model. In reality it possesses a more general character, and corresponds to a system of Fermi particles with arbitrary short-range forces having a radius of action small in comparison with the mean wavelength.

Such a model makes it possible to obtain expressions for the various quantities describing the Fermi fluid, for example the energy, the effective mass, and the function f, in the form of expansions in powers of $(a/\lambda\!\!\!^-)$ (a is the radius and $\lambda\!\!\!^-$ is the wavelength). We shall carry out the calculation to within terms of the order $(a/\lambda\!\!\!^-)^2$. By a procedure similar in principle one could determine also terms of several higher orders, but this is of no special interest. We begin with the calculation of the energy.

We use perturbation theory to take into account the interaction energy of the particles, which we write in the form (with the volume of the gas taken as unity for simplicity)

$$V = 2U \sum_{\substack{n_1 n_2 m_1 m_2 \\ n_1 < n_2}} a^+_{m_1} a^+_{m_2} a_{n_2} a_{n_1}, \tag{A1.1}$$

where a^+_i and a_i are creation and annihilation

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*For liquid He^4 in the visible region $h \sim 10^{-8}\text{cm}^{-1}\psi^{15}$ (about the same value should hold for He^3 for $\Delta\omega\tau \ll 1$). Although measurements of h for He^4 have been carried out, they are at the limit of what is experimentally possible.

operators for the particles. The summation is taken with the restriction that the total momentum is conserved; furthermore the spin component for the state m_1 is equal to that for the state n_1, and similarly for m_2 and n_2. The position of U outside the sign of summation corresponds to the fact that the interaction between all pairs of particles is the same, and the scattering amplitude does not depend on the angle. In first approximation the quantity U is connected with this amplitude by the relation

$$U = \frac{4\pi a\hbar^2}{m} \tag{A1.2}$$

(a is the scattering amplitude).

The first-order correction to the energy is the diagonal matrix element of V:

$$E_n^{(1)} = 2U \sum_{n_1<n_2} N_{n_1} N_{n_2} Q_{n_1 n_2}, \tag{A1.3}$$

where the N_i are occupation numbers.

The factor Q_{ik} in Eq. (A1.3) takes account of the fact that Fermi particles with a scattering amplitude independent of the angle do not interact in the case of parallel spins. Thus we shall take

$$Q_{ik} = \frac{1}{4} - \sigma_i \sigma_k \tag{A1.4}$$

(σ_i is the spin operator of the i-th particle).

Substituting Eqs. (A1.2) and (A1.4) in Eq. (A1.3), we get

$$E^{(1)} = (2\pi a\hbar^2/m)\, N^2/2. \tag{A1.5}$$

To find the second-order correction we use the expression from perturbation theory

$$E_n^{(2)} = \sum_{m\neq n} |V_{nm}|^2/(E_n - E_m). \tag{A1.6}$$

Substituting the expression (A1.1) in this formula, we get the following sum:

$$4U^2 \sum_{\substack{n_1 n_2 m_1 m_2 \\ n_1<n_2}} \frac{N_{n_1} N_{n_2} (1-N_{m_1})(1-N_{m_2})\, Q_{n_1 n_2}}{E_{n_1}+E_{n_2}-E_{m_1}-E_{m_2}}, \tag{A1.7′}$$

where N_i are the equilibrium occupation numbers and E_i the energies of the particles.

Because our purpose is to get the expansion of the energy in powers of a, we have to take into account the fact that the relation (A1.2) between U and the scattering amplitude is not exact, but is valid only to first order. When second-order terms are included we get instead of (A1.2) the following relations:

$$2U + 4U^2 \sum_{m_1 m_2} Q_{n_1 n_2}/(E_{n_1}+E_{n_2}-E_{m_1}-E_{m_2}) = \frac{8\pi a\hbar^2}{m}. \tag{A1.2′}$$

If we get from this the expression of U in terms of a and substitute it in Eq. (A1.3), the resulting expression for $E^{(1)}$ contains terms proportional to a^2, which must naturally be assigned to the second-order correction. Taking this into account, we get the following value of the second-order correction to the energy:

$$E^{(2)} = 2U^2 \sum_{n_1 n_2 m_1 m_2} \left\{ \frac{N_{n_1} N_{n_2} (1-N_{m_1})(1-N_{m_2})\cdot Q_{n_1} Q_{n_2}}{E_{n_1}+E_{n_2}-E_{m_1}-E_{m_2}} - \frac{N_{n_1} N_{n_2} Q_{n_1 n_2}}{E_{n_1}+E_{n_2}-E_{m_1}-E_{m_2}} \right\}. \tag{A1.7}$$

In view of the fact that the expression in brackets is symmetric in n_1 and n_2, we replace the restriction $n_1 > n_2$ by the factor 1/2.

The meaning of the operation just performed lies in the fact that in actuality an expansion in powers of U is not what is used. The existence of a constant U would simply lead to an infinity in the energy, as can be seen directly from the formula (A1.7). In the present case the essential point is that the scattering amplitude a has a finite, and moreover a small, value, which makes possible an expansion in powers of this quantity.

In the first term of the expression (A1.7) the term with the product of four N_i's is equal to zero because of the fact that the denominator is antisymmetric under the interchange $n_1 n_2 \rightleftarrows m_1 m_2$, while its numerator is symmetric, and all the ranges of summation are identical. The two surviving terms with products of three N_i's are identical. Thus we get finally:

$$E^2 = -4U^2 \sum_{n_1 n_2 m_1 m_2} \frac{N_{n_1} N_{n_2} N_{m_1} Q_{n_1 n_2}}{E_{n_1}+E_{n_2}-E_{m_1}-E_{m_2}}. \tag{A1.8}$$

This is the expression for the energy that is valid on the basic assumption $(a/\lambda) \ll 1$.

Our purpose is to obtain the characteristics of the degenerate Fermi gas. From the expression (A1.8) we find the energy of the ground state

$$E^{(2)} = -\frac{4U^2}{(2\pi\hbar)^3} \int\limits_{|\mathbf{p}_1|<p_0;} d\mathbf{p}_1 \int\limits_{|\mathbf{p}_2|<p_0;} d\mathbf{p}_2 \int\limits_{|\mathbf{p}_3|<p_0;} d\mathbf{p}_3 \int d\mathbf{p}_4 \frac{\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4)}{(\mathbf{p}_1^2+\mathbf{p}_2^2-\mathbf{p}_3^2-\mathbf{p}_4^2)\frac{1}{2m}}, \tag{A1.9}$$

where p_0 is the limiting momentum.

According to Sec. 1 the energy of the excitations is given by the relation

$$\varepsilon_i = \frac{\delta E}{\delta N_i}. \tag{A1.10}$$

Variation of the expressions (1.3) and (1.8) with respect to N_i gives

$$\varepsilon(\mathbf{p}) = \frac{\mathbf{p}^2}{2m} + \frac{UN}{2} + \frac{2U^2}{(2\pi\hbar)^9} \int d\mathbf{p}_1 \int d\mathbf{p}_2 \int d\mathbf{p}_3 \times \left[\frac{\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}-\mathbf{p}_3)}{(\mathbf{p}^2+\mathbf{p}_3^2-\mathbf{p}_1^2-\mathbf{p}_2^2)/2m} - 2\frac{\delta(\mathbf{p}_1+\mathbf{p}-\mathbf{p}_2-\mathbf{p}_3)}{(\mathbf{p}^2+\mathbf{p}_1^2-\mathbf{p}_2^2-\mathbf{p}_3^2)/2m} \right]. \tag{A1.11}$$

Thus the problem of calculating the energy of the ground state and the effective mass of the excitations reduces to the calculation of the integrals (A1.9) and (A1.11). The integration is rather cumbersome because of the high multiplicity of the integrals and the awkwardness of the regions of integration.

Instead of this we can use a simpler method, based on the introduction of the function f. If we introduce this function

$$f_{ik}=\delta^2E/\delta N_i\delta N_k, \qquad (A1.12)$$

then according to Eqs. (2.5) and (8.7) we can determine the effective mass and the speed of low-frequency sound. By a suitable integration we can find the energy of the ground state from the speed of sound.

Thus the problem reduces to the determination of the quantity f. Taking the variations of the expressions (A1.3) and (A1.8) with respect to N_1, and then with respect to N_k, we find the following expression:

$$f=2UQ_{\sigma,\sigma'}-\frac{8U^2}{(2\pi h)^3}\int d\mathbf{p}_1\int\limits_{|\mathbf{p}_1|<p_0} d\mathbf{p}_2$$
$$\times\left[Q_{\sigma\sigma'}\frac{\delta(\mathbf{p}+\mathbf{p}'-\mathbf{p}_1-\mathbf{p}_2)}{(\mathbf{p}^2+\mathbf{p}'^2-\mathbf{p}_1^2-\mathbf{p}_2^2)/2m}\right. \qquad (A1.13)$$
$$\left.+\frac{1}{4}\frac{\delta(\mathbf{p}+\mathbf{p}_1-\mathbf{p}'-\mathbf{p}_2)}{(\mathbf{p}^2+\mathbf{p}_1^2-\mathbf{p}'^2-\mathbf{p}_2^2)/2m}+\frac{1}{4}\frac{\delta(\mathbf{p}'+\mathbf{p}_1-\mathbf{p}-\mathbf{p}_2)}{(\mathbf{p}'^2+\mathbf{p}_1^2-\mathbf{p}^2-\mathbf{p}_2^2)/2m}\right].$$

In the calculation we shall at once set $|\mathbf{p}| = |\mathbf{p}_1| = p_0$. The integration in the second term of f is considerably simpler than those in Eqs. (A1.9) and (A1.11). We get as the result:

$$f(\chi)=\frac{2\pi a\hbar^2}{m}\left[1+2\left(\frac{3}{\pi}\right)^{\frac{1}{3}}aN^{\frac{1}{3}}\left(2+\frac{\cos\chi}{2\sin\frac{\chi}{2}}\ln\frac{1+\sin\frac{\chi}{2}}{1-\sin\frac{\chi}{2}}\right)\right]$$
$$-\frac{8\pi a\hbar^2}{m}(\sigma_1\sigma_2)\left[1+2\left(\frac{3}{\pi}\right)^{\frac{1}{3}}aN^{\frac{1}{3}}\left(1-\frac{\sin\frac{\chi}{2}}{2}\right)\ln\frac{1+\sin\frac{\chi}{2}}{1-\sin\frac{\chi}{2}}\right].$$
$$(A1.14)$$

A peculiarity of this expression deserves attention. For angles χ close to π the function f has a logarithmic singularity,

$$f(\chi)\sim\left(\frac{1}{4}-\sigma_1\sigma_2\right)\ln\frac{1}{\pi-\chi}. \qquad (A1.15)$$

It is clear that in this case the approximation we have used is, strictly speaking, not legitimate. The singularity of the function f at $\chi = \pi$ is a manifestation of the singularity in the scattering amplitude of excitations colliding at the angle π. The corresponding expression, obtained by summing the main terms of the perturbation theory, is proportional to

$$\left[1+2\left(\frac{3}{\pi}\right)^{\frac{1}{3}}aN^{\frac{1}{3}}\ln\frac{p_0^2}{\varepsilon}\right]^{-1}, \qquad (A1.16)$$

where $\epsilon = p^2 + p'^2 - 2p_0^2$.

In the case of positive a this expression goes to zero for $p^2 = p'^2 = p_0^2$.

If, however, a < 0 (this is possible for Fermi systems), the scattering amplitude has a pole near the Fermi surface. This corresponds to the possibility of the formation of bound pairs of excitations with opposite momenta, which was pointed out by Cooper[19] and is the main cause of the phenomenon of superconductivity in metals.[20]

Thus the expression found for f is not valid for angles close to π. But in view of the fact that the singularity is only logarithmic, it manifests itself only in the immediate neighborhood of the singular point. And since only integrals of f together with regular functions occur in the expressions with which we are concerned, the logarithmic singularity of the function f is of no importance.

Substituting Eq. (A1.4) in Eq. (2.5), we find the value of the effective mass

$$m/m^*=1-(8/15)(3/\pi)^{\frac{2}{3}}(7\ln 2-1)a^2N^{\frac{2}{3}}. \qquad (A1.17)$$

We note that if we insert here the value m* = 1.43 m, which corresponds to liquid He^3, the resulting value of a is 1.6×10^{-8}; that is, it is of the order of magnitude of the gas-kinetic value of the diameter of the helium atom. Such a comparision of course has no meaning in a strict sense. The model under consideration cannot describe liquid He^3. This can already be seen from the fact that the quantity (m* − m)/m*, which in the theory should be a quantity of second order in a, is 1/3 for He^3.

Substituting the formula for f in the expression for the speed of sound, we get:

$$(A1.18)$$
$$c^2=\frac{\pi^{\frac{4}{3}}}{3^{\frac{1}{3}}}N^{\frac{2}{3}}\frac{\hbar^2}{m^2}+2\frac{\pi ah^2}{m^2}N\left[1+\frac{4}{15}\left(\frac{3}{\pi}\right)^{\frac{1}{3}}aN^{\frac{1}{3}}(11-2\ln 2)\right].$$

From the value of c^2 just found it is not hard to obtain the energy of the ground state of the Fermi fluid. To do this we use the relation (8.2), $c^2 = N/m(\partial\mu/\partial N)$. This gives:

$$(A1.19)$$
$$E=\int\mu\, dN=E^{(0)}+\frac{\pi a\hbar^2}{m}N^2\left[1+\frac{6}{35}\left(\frac{3}{\pi}\right)^{\frac{1}{3}}aN^{\frac{1}{3}}(11-2\ln 2)\right].$$

Equation (A1.2) agrees with the result of Lee and Yang.[15]

The same result can also be obtained by direct integration in Eq. (A1.9). To calculate the integral in Eq. (A1.9) it is convenient to introduce new variables

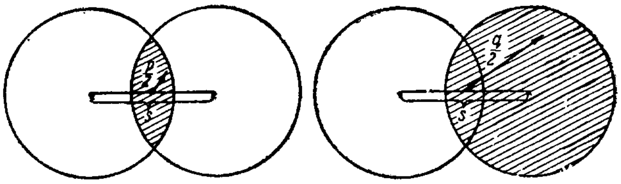

FIG. 2

$$\mathbf{p}=\mathbf{p}_1-\mathbf{p}_2;\quad \mathbf{q}=\mathbf{p}_3-\mathbf{p}_4;\quad \mathbf{s}=\mathbf{p}_1+\mathbf{p}_2=\mathbf{p}_3+\mathbf{p}_4\cdot$$

In these variables $E^{(2)}$ takes the form

$$E^{(2)}=-\frac{mU^2}{4\,(2\pi h)^2}\int d\mathbf{s}\int d\mathbf{p}\int d\mathbf{q}\,\frac{1}{p^2-q^2}\,,$$

where the region of integration for the vector s is $0 < |\mathbf{s}| < 2p_0$, and the regions of values for the vectors **p** and **q** are shown in Fig. 2. If we introduce the variables $x_q = \cos(\mathbf{q}, \mathbf{s})$ and $x_p = \cos(\mathbf{p}, \mathbf{s})$, $E^{(2)}$ is given by

$$E^{(2)}=-\frac{8mU^2\pi^3}{(2\pi h)^9}\int\limits_0^{2p_0} s^2\,ds\int\limits_0^1 dx_p\int\limits_0^{z(x_p)} p^2\,dp\int\limits_{-1}^1 dx_q\int\limits_0^{z(x_q)} q^2\,dq\,\frac{1}{p^2-q^2}\,,$$

where z(x) satisfies the relation

$$z^2+2zsx+s^2=4p_0^2.$$

By a series of transformations and integrations by parts with respect to dx_p and dx_q we then find:

$$E^2=\frac{2mU^2p_0^7}{\pi^6h^9}\int\limits_0^1 s^2\,ds\Bigg[\int\limits_0^{1+s} p^2\,dp\int\limits_0^{1-s} q^2\,dq$$

$$+\frac{1}{4s^2}\int\limits_{\sqrt{1-s^2}}^{1+s} p\,dp\,(1-p^2-s^2)\int\limits_{\sqrt{1-s^2}}^{1-s} q\,dq\,(1-q^2-s^2)\Bigg]\frac{1}{p^2-q^2}\,.$$

Proceeding further to integrate by parts with respect to ds and then performing the remaining integration, we get:

$$E^{(2)}=(6/35)\,(3/\pi)^{\frac{1}{3}}\,(11-2\ln 2)\,aN^{\frac{1}{3}}E^{(1)}.$$

Here we have inserted the expression (A1.2) for U and have set $p_0 = \hbar(3\pi^2N)^{1/3}$. This result is identical with the second-order term in Eq. (A1.19).

A.2. Microscopic Theory of the Fermi Fluid for T = 0

Here we present the general microscopic theory of the Fermi fluid at T = 0, as developed recently by Landau;[18] this theory is valid for arbitrary interactions. As we shall see, this treatment provides a basis for the phenomenological theory and an understanding of the physical meaning of the function f that appears in it.

The microscopic treatment is based on methods taken from the quantum theory of fields. As is well known, in quantum field theory one works with vacuum expectation values of chronological products of field operators (i.e., products of operators taken in the order of succession of their times). These quantities are the so-called Green's functions. In our case, instead of vacuum expectation values we shall consider expectation values for the ground state of the system. It is not hard to verify that such expectation values have the same properties as the Green's functions of the field theory, and in particular that in principle they can be calculated by means of the Feynman-diagram technique.

For definiteness we shall first give the name of Green's function to the quantity

$$G_{1,\,2}=-i\,\langle T\,(\psi_1\psi_2^+)\rangle, \qquad \text{(A2.1)}$$

where the brackets < > denote the expectation value for the ground state, ψ and ψ^+ are the field operators of the Fermi particles, the indices 1, 2 denote sets of space, time, and spin coordinates, and T, as usual, denotes the time-ordered product with change of sign when ψ_1 and ψ_2^+ are interchanged. In the absence of external fields the Green's function depends only on the differences of the coordinates and the times. As for the spin indices, if the interaction of the particles does not depend on their spins, then $G_{\alpha\beta} \sim \delta_{\alpha\beta}$.

We denote the Fourier transform of the Green's function with respect to the coordinate and time differences by $G(p)\,\delta_{\alpha\beta}$ [here p is the four-dimensional vector $(\mathbf{p}, \epsilon)$]. The poles of G(p) determine the energies of the quasi-particles. If the magnitude of **p** is close to p_0 and ϵ is close to the limiting energy μ, then G(p) has the form

$$G(p)=\frac{a}{\varepsilon-\mu-v\,(p-p_0)+i\delta}\,. \qquad \text{(A2.2)}$$

Here a is a positive constant, and δ is a small quantity which agrees in sign with the sign of $\epsilon - \mu$, or, close to a pole, with the sign of $p - p_0$ (δ is introduced to fix the rule for going around the poles, so that it will correspond to taking the T-product in the **x**, t representation).

Let us now introduce the expectation value of the time-ordered product of four operators

$$\Phi_{12,\,34}=\langle T\,(\psi_1\psi_2\psi_3^+\psi_4^+)\rangle. \qquad \text{(A2.3)}$$

We give the name of vertex part $\Gamma_{12,34}$ to the quantity defined by the following relation:

$$\Phi_{12,\,34}=G_{1,3}\,G_{2,4}-G_{1,4}\,G_{2,3}$$
$$+i\sum_{1',\,2',\,3',\,4'}G_{1,1'}\,G_{2,2'}\,G_{3,3'}\,G_{4,4'}\,\Gamma_{1'2',3'4'}. \qquad \text{(A2.4)}$$

The Fourier transform of the vertex part has the form

$$(2\pi h)^4\,\Gamma_{\alpha\beta,\,\gamma\delta}(p_1p_2,\ p_3p_4)\,\delta(p_1+p_2-p_3-p_4).$$

It is clear that the function $\Gamma_{\alpha\beta\ \gamma\delta}$ (P_1P_2, P_3P_4) changes sign on the interchange $\alpha p_1 \rightleftarrows \beta p_2$. As in the quantum field theory, the function Γ with a definite relation between the components of each of the vectors P [for example, near the Fermi surface $\epsilon - \mu = v(p - p_0)$], multiplied by a^2, plays the role of the scattering amplitude of the quasi-particles.

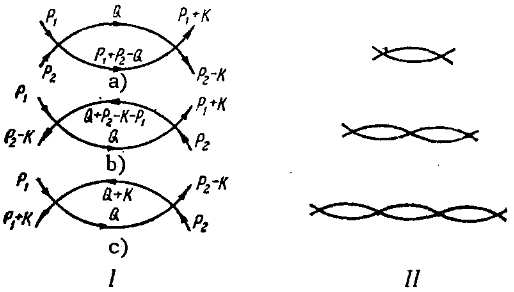

FIG. 3

Let us consider Γ in the case in which p_1 is close to p_3, and consequently p_2 is close to p_4, and introduce the definition

$$\Gamma(p_1p_2,\ p_1+K\ p_2-K)\equiv\Gamma(P_1P_2,\ K),$$

where $K = (\mathbf{k}, \omega)$ is a small four-vector.

If it were possible to calculate such a vertex part by perturbation theory, then in first order one would have to take the diagrams shown in Figs. 3, I. The corresponding expressions contain integrals of products of two Green's functions. Whereas for diagrams (a) and (b) the case $K = 0$ is not in any way singled out, in case (c) the poles of the two Green's functions approach each other for $K = 0$, which, as we shall see, leads to the appearance of a singularity of Γ.

If we denote by $\Gamma^{(1)}$ the set of all possible diagrams for Γ that do not have "singular elements" [i.e., integrals of $G(p)G(p) + K)$], it is not hard to see that the total Γ is obtained by summation of the "ladders" shown in Fig. 3, II. This can be accomplished by solving the integral equation

$$\Gamma_{\alpha\beta,\,\gamma\delta}(P_1P_2;\ K)=\Gamma^{(1)}_{\alpha\beta,\,\gamma\delta}(P_1P \qquad \text{(A2.5)}$$
$$\frac{i}{(2\pi\hbar)^4}\int\Gamma^{(1)}_{\alpha\varepsilon,\,\gamma\zeta}(P_1Q)\,G(Q)\,G(Q+K)\,\Gamma_{\zeta\beta,\,\varepsilon\delta}(QP_2;\ K)\,d^4Q.$$

Here, in view of the fact that $\Gamma^{(1)}$ has no singularity at $K = 0$, we simply set $K = 0$ everywhere in $\Gamma^{(1)}$.

Let us now examine the product in the integrand. The integral over Q will consist of the principal value and a term arising from the passage around the poles. Because the arguments of the two G functions are nearly equal, we can assume that all the other parts of the integral have the same values at these poles. The passage around the poles then gives a contribution only in the case in which the detours are on opposite sides of the real axis. This means either $q < p_0$, $|\mathbf{q} + \mathbf{k}| > p_0$, or vice versa. Because of the smallness of $\mathbf{k}$ it is not hard to see that here $q \approx p_0$ and $\epsilon_q \approx \mu$. Thus in the part of the integral over Q which comes from the passage around the poles the product $G(Q)G(Q + K)$ can be replaced by $A\,\delta(\epsilon - \mu)\,\delta(q - p_0)$.

The coefficient A can be found by integrating $G(Q)G(Q + K)$, and is found to be $(2\pi i a^2 k \cos\theta)/(\omega - vk\cos\theta)$, where θ is the angle between $\mathbf{q}$ and $\mathbf{k}$. Thus the product $G(Q)G(Q + K)$ can be written in the form

$$G(Q)\,G(Q+K)=2\pi i a^2\frac{k\cos\theta}{\omega-vk\cos\theta}\,\delta(\varepsilon-\mu)\,\delta(q-p_0)+\varphi(Q), \qquad \text{(A2.6)}$$

where φ corresponds to the principal value of the integral and has no singularities (therefore one puts $K = 0$ in this term).

If we consider the expression (A2.6) in the limit $K = 0$, then it makes a great difference how this limit is taken, i.e., what the ratio of ω and k approaches. The same is true about Γ in the limit $\omega \to 0$, $\mathbf{k} \to 0$.

First of all let us consider Γ in the limit $\mathbf{k} \to 0$, $(k/\omega) \to 0$. For this limit, which we denote by Γ^{ω}, we get from Eqs. (A2.5) and (A2.6):

$$\Gamma^{\omega}_{\alpha\beta,\,\gamma\delta}(P_1P_2)=\Gamma^{(1)}_{\alpha\beta,\,\gamma\delta}(P_1P_2) \qquad \text{(A2.7)}$$
$$-\frac{i}{(2\pi\hbar)^4}\int\Gamma^{(1)}_{\alpha\xi,\,\gamma\zeta}(P,\ Q)\,\varphi(Q)\,\Gamma^{(\omega)}_{\zeta\beta,\,\varepsilon\delta}(QP_2)\,d^4Q.$$

It is not hard to eliminate $\Gamma^{(1)}$ from the two equations (A2.5) and (A2.7). Using Eq. (A2.6), we find after simple transformations

$$\Gamma_{\alpha\beta,\,\gamma\delta}(P_1P_2;\ K)=\Gamma^{(\omega)}_{\alpha\beta,\,\gamma\delta}(P_1P_2) \qquad \text{(A2.8)}$$
$$+\frac{a^2p_0^2}{(2\pi\hbar)^3}\int\Gamma^{\omega}_{\alpha\varepsilon,\,\gamma\zeta}(P,\ Q)\,\Gamma_{\zeta\beta,\,\varepsilon\delta}(QP_2,\ K)\,\frac{k\cos\theta}{\omega-vk\cos\theta}\,d\Omega.$$

Let us now take the other limit, namely $K \to 0$, $(\omega/k) \to 0$. This limiting value, multiplied by a^2, corresponds physically to the forward scattering amplitude for quasi-particles with energy on the Fermi surface. We denote this quantity by Γ^k. From Eq. (A2.8) we find the relation between $a^2\Gamma^k$ and $a^2\Gamma^{\omega}$

$$a^2\Gamma^{k}_{\alpha\beta,\,\gamma\delta}(P_1P_2)=a^2\Gamma^{\omega}_{\alpha\beta,\,\gamma\delta}(P_1P_2) \qquad \text{(A2.9)}$$
$$-\frac{p_0^2}{v(2\pi\hbar)^3}\int a^2\Gamma^{\omega}_{\alpha\varepsilon,\,\gamma\zeta}(P,\ Q)\,a^2\Gamma^{k}_{\zeta\beta,\,\varepsilon\delta}(QP_2)\,d\Omega.$$

Let us now study the poles of the function $\Gamma(P_1P_2;\ K)$. Just as the poles of the function $G(P)$ give the relation between the energies and momenta of the Fermi excitations, the poles of the function Γ characterize the "two-particle" excitations, in other words the Bose excitations. In view of the fact that in the neighborhood of a pole $\Gamma(P_1 P_2; K) \gg \Gamma^{\omega}(P_1 P_2)$, we can neglect the term Γ^{ω} in the right member. Furthermore it must be noted that the variable P_2, and also the indices β and δ play the part of parameters in the equation. The function Γ can be represented in the form of a product $\chi_{\alpha\beta}(P_1; K)\, \chi_{\beta\delta}(P_2)$, and $\chi(P_2)$ cancels on the two sides of the equation. If we also introduce the notation

$$\nu_{\alpha\gamma}(\mathbf{n}) = \frac{\mathbf{n}\cdot\mathbf{k}}{\omega - v\mathbf{n}\cdot\mathbf{k}}\, \chi_{\alpha\gamma}(P_1; K),$$

where $\mathbf{n}$ is the unit vector in the direction of P_1, we get for $\nu_{\alpha\gamma}$ the equation

$$(\omega - v\mathbf{n}\cdot\mathbf{k})\,\nu_{\alpha\gamma}(\mathbf{n}) = \mathbf{n}\cdot\mathbf{k}\, \frac{p_0^2 a^2}{(2\pi\hbar)^3} \int \Gamma^{\omega}_{\alpha\varepsilon,\,\gamma\xi}(\mathbf{n}, \mathbf{l})\, \nu_{\xi\varepsilon}(\mathbf{l})\, d\Omega. \quad \text{(A2.10)}$$

Comparing this equation with Eqs. (8.14) and (10.4), we see without difficulty that it is the same as the equation for the zeroth sound and the spin waves, with the quantity $a^2\Gamma^{\omega}$ playing the role of the function f. This confirms the equation for the zeroth sound obtained in the phenomenological theory, and thus confirms the hypothesis made in that theory about the functional dependence of the energy of the excitations on the distribution function. On the other hand, owing to Eq. (A2.9), we have established a connection between the function f and the scattering amplitude of the quasi-particles at the angle 0°.

Let us examine this relation in greater detail. Denoting the amplitude multiplied by $(d\tau/d\epsilon)_{\epsilon=\mu}$ by $A(\mathbf{n}_1\cdot\sigma_1, \mathbf{n}_2\cdot\sigma_2)$, we get:

$$A(\mathbf{n}_1\cdot\sigma_1, \mathbf{n}_2\cdot\sigma_2) = \Phi(\mathbf{n}_1\cdot\sigma_1, \mathbf{n}_2\cdot\sigma_2) - \frac{1}{2}\operatorname{Sp}_{\sigma}' \int \Phi(\mathbf{n}_1\cdot\sigma_1, \mathbf{n}'\cdot\sigma')\, A(\mathbf{n}'\cdot\sigma', \mathbf{n}_2\cdot\sigma_2)\, \frac{d\Omega}{4\pi}. \quad \text{(A2.11)}$$

As regards their spin dependence, Φ and A each contain two terms: one independent of the spins, and one proportional to $\sigma_1\cdot\sigma_2$. If we write

$$A(\mathbf{n}_1\cdot\sigma_{21}, \mathbf{n}\cdot\sigma_2) = B(\mathbf{n}_1\cdot\mathbf{n}) + \sigma_1\cdot\sigma_2\cdot C(\mathbf{n}_1, \mathbf{n}_2), \quad \text{(A2.12)}$$

B and C satisfy the equations

$$B(\mathbf{n}_1, \mathbf{n}_2) = F(\mathbf{n}_1, \mathbf{n}_2) - \int F(\mathbf{n}_1, \mathbf{n}')\, B(\mathbf{n}', \mathbf{n}_2)\frac{d\Omega}{4\pi}, \quad \text{(A2.13)}$$

$$C(\mathbf{n}_1, \mathbf{n}_2) = Z(\mathbf{n}_1, \mathbf{n}_2) - \frac{1}{4}\int Z(\mathbf{n}_1, \mathbf{n}')\, C(\mathbf{n}', \mathbf{n}_2)\frac{d\Omega}{4\pi}.$$

Being scalars, the quantities B and C, and also F and Z, can depend only on cos χ. If we introduce expansions in spherical harmonics, $B(\chi) = \Sigma B_l P_l(\cos\chi)$, and so on, then we can easily get relations between the coefficients B_l and F_l, and between C_l and Z_l:

$$B_l = \frac{F_l}{1 + \frac{F_l}{2l+1}}, \quad C_l = \frac{Z_l}{1 + \frac{Z_l}{4(2l+1)}}. \quad \text{(A2.14)}$$

For example, let us find the scattering amplitude of the excitations for a rarefied Fermi gas. From Eqs. (A1.16) and (A2.13) we get:

$$A(\chi, \sigma_1\cdot\sigma_2) = 8\left(\frac{3}{\pi}\right)^{\frac{1}{3}} aN^{\frac{1}{3}} \left(\frac{1}{4} - \sigma_1\cdot\sigma_2\right) \quad \text{(A2.15)}$$

$$+ 16\left[\left(\frac{3}{\pi}\right)^{\frac{1}{3}} aN^{\frac{1}{3}}\right]^2 \left[\frac{1}{4} - \sigma_1\cdot\sigma_2 + \frac{\cos\chi}{8\sin\frac{\chi}{2}} \ln\frac{1+\sin\left(\frac{\chi}{2}\right)}{1-\sin\left(\frac{\chi}{2}\right)} - \sigma_1\cdot\sigma_2\left(1 - \frac{\sin\left(\frac{\chi}{2}\right)}{2}\ln\frac{1+\sin\left(\frac{\chi}{2}\right)}{1-\sin\left(\frac{\chi}{2}\right)}\right)\right].$$

We note that in the case of parallel spins this expression goes to zero for $\chi = 0$, as it must according to the Pauli principle (for f this does not occur).

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[1] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **30**, 1058 (1956), Soviet Phys. JETP **3**, 920 (1956).

[2] I. M. Khalatnikov and A. A. Abrikosov, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 915 (1957), Soviet Phys. JETP **5**, 745 (1957).

[3] Abraham, Osborne, and Weinstock, Phys. Rev. **98**, 551 (1955).

[4] Fairbank, Ard, and Walters, Phys. Rev. **95**, 567 (1954).

[5] A. A. Abrikosov and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 1083 (1957), Soviet Phys. JETP **5**, 887 (1957).

[6] I. Ia. Pomeranchuk, J. Exptl. Theoret. Phys. (U.S S.R.) **20**, 919 (1950).

[7] K. N. Zinov'eva, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 609 (1958), Soviet Phys. JETP **7**, 421 (1958).

[8] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 59 (1957), Soviet Phys. JETP **5**, 101 (1957).

[9] Lagner, Sydoriak, and Roberts, Program of the Fifth International Conference on Low-Temperature Physics and Chemistry, p. 41. University of Wisconsin 1957.

[10] I. M. Khalatnikov and A. A. Abrikosov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 110 (1957), Soviet Phys. JETP **6**, 84 (1958).

[11] A. A. Abrikosov and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **34**, 198 (1958), Soviet Phys. JETP **7**, 135 (1958).

[12] L. D. Landau and E. M. Lifshitz, Электродинамика сплошных сред (Electrodynamics of Continuous Media), Gostekhizdat, Moscow 1957.

[13] S. M. Rytov. Теория электрических флуктуаций

и теплового излучения (Theory of Electrical Fluctuations and Thermal Radiation), Acad. Sci. Press, 1953.

[14] L. D. Landau and E. M. Lifshitz, J. Exptl. Theoret. Phys. (U.S.S.R.) **32**, 618 (1957), Soviet Phys. JETP **5**, 512 (1957).

[15] V. L. Ginzburg, J. Exptl. Theoret. Phys. (U.S.S.R.) **13**, 243 (1943).

[16] A. A. Abrikosov and I. M. Khalatnikov, J. Exptl. Theoret. Phys. (U.S.S.R.) **33**, 1154 (1957), Soviet Phys. JETP **6**, 888 (1958).

[17] K. Huang and C. N. Yang, Phys. Rev. **105**, 767 (1957).

[18] T. D. Lee and C. N. Yang, Phys. Rev. **105**, 1119 (1957).

[19] L. N. Cooper, Phys. Rev. **104**, 1189 (1956).

[20] Bardeen, Cooper, and Schrieffer, Phys. Rev. **106**, 162 (1957).

[21] L. D. Landau, J. Exptl. Theoret. Phys. (U.S.S.R.) (in press).

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