Quasiparticle Lifetime in a Finite System: A Nonperturbative Approach

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The problem of electron-electron lifetime in a quantum dot is studied beyond perturbation theory by mapping onto the problem of localization in the Fock space. Localized and delocalized regimes are identified, corresponding to quasiparticle spectral peaks of zero and finite width, respectively. In the localized regime, quasiparticle states are single-particle-like. In the delocalized regime, each eigenstate is a superposition of states with very different quasiparticle content. The transition energy is $\epsilon_c \simeq \Delta(g/\ln g)^{1/2}$, where Δ is mean level spacing, and g is the dimensionless conductance. Near ϵ_c there is a broad critical region not described by the golden rule. [S0031-9007(97)02895-0]

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Quasiparticle in a Fermi liquid is not an eigenstate: it decays into two quasiparticles and a hole. In an infinite clean system, by using the golden rule (GR), quasiparticle decay rate is estimated as $\gamma(\epsilon) \sim \epsilon^2/\epsilon_F$, where ϵ is quasiparticle energy and ϵ_F is Fermi energy [1]. However, in a finite system the eigenstate spectrum is discrete. In this case, quasiparticles may be viewed as wave packets constructed of such states, the packet width being determined by the lifetime in an infinite system: $\delta\epsilon \simeq \gamma(\epsilon)$. In this paper we attempt to clarify the relation between quasiparticles and many-particle states, and find that at different energies it has different meanings.

Conventionally, quasiparticles are well defined provided $\gamma(\epsilon) \ll \epsilon$. However, to resolve quasiparticles in a mesoscopic system, a more stringent condition is required: $\gamma(\epsilon) < \Delta$, the quasiparticle level spacing. As an example, consider quasiparticle peaks in tunneling conductance of a quantum dot [2,3]. The peaks observed in nonlinear conductance at certain bias are interpreted as the quasiparticle tunneling density of states (DOS). Each peak corresponds to a "quasiparticle state," and its width measures the lifetime of the state. Below we consider an isolated Fermi liquid, ignoring any contributions to the quasiparticle decay due to finite escape rate, phonons, etc. [4].

The meaning of quasiparticle lifetime needs clarification: strictly speaking, since a quantum dot is a finite system, any many-particle eigenstate gives rise to an infinitely narrow conductance peak. However, we will see that only a small fraction of those states overlap with one-particle excitations strongly enough to be detected by a finite sensitivity measurement. Under certain conditions, these strong peaks group into clusters of the width $\sim\gamma(\epsilon)$ that can be interpreted as quasiparticle peaks.

Before discussing possible regimes let us review the GR approach. Recently Sivan *et al.* [5], adopting the quasiparticle picture to a finite size geometry and relying on the earlier work [6] on electron-electron scattering rate in diffusive conductors, found that

$$\gamma(\epsilon) \approx \Delta(\epsilon/g\Delta)^2, \quad \epsilon < g\Delta\,, \tag{1}$$

where Δ is the mean single-particle level spacing near the Fermi level and $g \gg 1$ is the dimensionless conductance, for a finite system defined by $g = E_c/\Delta$, where E_c is the Thouless energy (inverse time of diffusion through the system). The decay rate (1) is much larger than in a clean Fermi liquid; however, at $\epsilon < E_c$ one has $\gamma(\epsilon) \ll \Delta$, implying that the quasiparticle states can be resolved.

However, the GR can be used to evaluate lifetime only when the density of final states is sufficiently large, so that the GR decay rate is larger than the level spacing of final unperturbed states. Otherwise, the GR will not give the decay rate, but rather just a first-order perturbation correction to the energy of a given eigenstate. In our problem, it is important to realize that, since a quasiparticle decays into three quasiparticles, the density of relevant final states, $\nu_3(\epsilon) = \epsilon^2/2\Delta^3$, is much smaller than that of all many-body states. The interaction matrix element V in the GR leading to Eq. (1) is of the order of Δ/g (see below), which should be compared to the three-particle level spacing $1/\nu_3$. Therefore, the GR is not applicable unless $\epsilon > \epsilon^* = \Delta\sqrt{g}$. Note that, since $\epsilon^* \gg \Delta$, there are many states whose lifetime is not given by GR.

At $\epsilon \ll \epsilon^*$, when matrix elements are smaller than the spacing $1/\nu_3$, the quasiparticle states do not decay: they are just slightly perturbed one-particle states. Hence they produce narrow conductance peaks that may have weak satellites due to coupling to many-particle states. As ϵ approaches ϵ^* from below, the number of the satellites rapidly increases. At $\epsilon \gg \epsilon^*$, clusters of satellites form finite width peaks well described by the GR.

For a quantitative description of the interval $0 < \epsilon < E_c$ (including the vicinity of ϵ^*), it is both interesting and instructive to explore the analogy of this problem with the Anderson localization. This is the goal of our paper.

Extension of the traditional localization problem to few interacting particles has received much attention recently. The study of the two-particle case, started by Dorokhov [7], was further advanced by Shepelyansky [8], Imry [9], and Pichard *et al.* [10], with extensions to more

particles. Some of the energy scales encountered below, such as V and $1/\nu_3$, have already been discussed in the context of those papers. In contrast, here we deal with the states extended throughout the whole finite system. The localization we consider occurs in the Fock space of many-body states, rather than in the real space.

Distance in Fock space.—This is a concept we introduce to measure closeness of many-body states. Consider a generic two-body interaction Hamiltonian in a secondary quantized form

$$\mathcal{H}_0 + \mathcal{H}_1 = \sum_\alpha \epsilon_\alpha c_\alpha^\dagger c_\alpha + \sum_{\alpha\beta\gamma\delta} V_{\gamma\delta}^{\alpha\beta} c_\gamma^\dagger c_\delta^\dagger c_\beta c_\alpha . \quad (2)$$

The many-body problem is formulated in the Fock space, by choosing as a basis $\{\Psi_N\}$—the Slater determinants constructed out of the N-particle Fermi vacuum $|N\rangle$,

$$\Psi_N = c_{\alpha_{2m}}^\dagger \cdots c_{\alpha_{m+1}}^\dagger c_{\alpha_m} \cdots c_{\alpha_1} |N\rangle . \quad (3)$$

Any state Ψ_N can be represented as a string with entries 1 and 0 labeling the single-particle states which do or do not participate in Ψ_N, respectively. Let us define the distance between two states Ψ_N and Ψ_N' as the number of positions in which the corresponding strings differ.

Thus defined, the distance in the Fock space can be used to map the lifetime problem to an appropriate Anderson localization problem. It is useful to think of the states Ψ_N, the eigenstates of $\mathcal{H}_0$, as "site orbitals" in the Fock space, each having an on-site energy $\epsilon_{\alpha_{2m}} + \cdots + \epsilon_{\alpha_{m+1}} + |\epsilon_{\alpha_m}| + \cdots + |\epsilon_{\alpha_1}|$. These sites are interconnected by the interaction $\mathcal{H}_1$, which we think of as hopping in the Anderson problem (the diagonal part of $\mathcal{H}_1$ is added to $\mathcal{H}_0$ by using the Hartree-Fock method). The point is that the two-body interaction matrix element $\langle\Psi_N|\mathcal{H}_1|\Psi_N'\rangle$ is nonzero only if the distance between the states Ψ_N and Ψ_N' equals 0, 2, or 4. We construct a network in the Fock space by connecting all orbitals Ψ_N which are at a distance 2 from each other.

Below we study localization on this network. We show that there is a critical energy ϵ^{**} above which the GR is relevant and the states are extended, whereas at lower energies the states are localized and the GR breaks down. Above the localization threshold (and beyond the critical region) we recover the GR picture [5] with finite width quasiparticle conductance peaks.

The meaning of localization in the Fock space is that a localized state is practically identical to a single-particle excitation (or a superposition of very few quasiparticle states). The energy of each of the constituent quasiparticles represents a good quantum number, whereas for the extended delocalized states only total energy is conserved. The transition is of the Anderson type because the two-body Hamiltonian is *local* in the Fock space: it couples only the orbitals of similar quasiparticle content.

It is worth remarking that the hierarchy of many-particle states proved to be a useful picture in the compound nucleus theory [11]. The "doorway states" introduced in these studies, although they serve a different purpose, are related to our network construction.

Hopping over the network in the Fock space.—The two-body interaction matrix elements in (2) are given by

$$V_{\gamma\delta}^{\alpha\beta} = \iint dx\, dx'\, V(x - x')\psi_\delta^*(x)\psi_\gamma^*(x')\psi_\beta(x)\psi_\alpha(x') . \quad (4)$$

To evaluate the matrix elements, let us consider diffusive disorder and a short range interaction, $V(x - x') = \lambda\Delta\mathcal{V}\delta(x - x')$, where $\mathcal{V}$ is the volume, and $\lambda \approx 1$ is the dimensionless interaction strength. For $\alpha \neq \beta \neq \gamma \neq \delta$, $V_{\gamma\delta}^{\alpha\beta}$ is a random quantity with zero average. The root-mean-square V may be evaluated [12], e.g., by using the diagram shown in Fig. 1(a). In the absence of time reversal symmetry we obtain

$$V = \lambda b_d \frac{\Delta^2}{E_c} ; \quad b_d^2 = \frac{2}{\pi^2} \sum_{m\neq 0} \frac{\gamma_1^2}{\gamma_m^2} , \quad (5)$$

where γ_m are eigenvalues of the diffusion operator. (By definition, $E_c = \gamma_1$.) In deriving (5) we assume small single-particle energies: $\epsilon_{\alpha(\beta,\gamma,\delta)} \leq E_c$. The magnitude of V decreases algebraically when the differences between the single-particle energies exceed E_c; below we shall ignore such contributions.

The network in the Fock space organizes all states in a hierarchy. Let $|N - 1\rangle$ be the ground state of $N - 1$ particles. The states $\Upsilon^\alpha = c_\alpha^\dagger|N - 1\rangle$ representing one particle added in the state α form the first generation of the hierarchy [see Fig. 1(b)]. The states $\Upsilon_\gamma^{\alpha\beta} = c_\alpha^\dagger c_\beta^\dagger c_\gamma|N - 1\rangle$, representing two particles and one hole, form the generation 3. Similarly, the generation 5 is formed by $\Upsilon_{\lambda\mu}^{\alpha\beta\gamma} = c_\alpha^\dagger c_\beta^\dagger c_\gamma^\dagger c_\lambda c_\mu|N - 1\rangle$, etc. The two-body interaction $\mathcal{H}_1$ couples only the states of near generations, so that any given state from generation $2n - 1$ is connected only to states from generations $2n + 1$, $2n - 1$, or $2n - 3$. This implies that connected states are a distance 2 from each other.

Consider now a state of generation 1, with an on-site energy ϵ. The DOS in generation 3 accessible by "hopping," having the same energy, is $\nu_3(\epsilon) = \epsilon^2/2\Delta^3$.

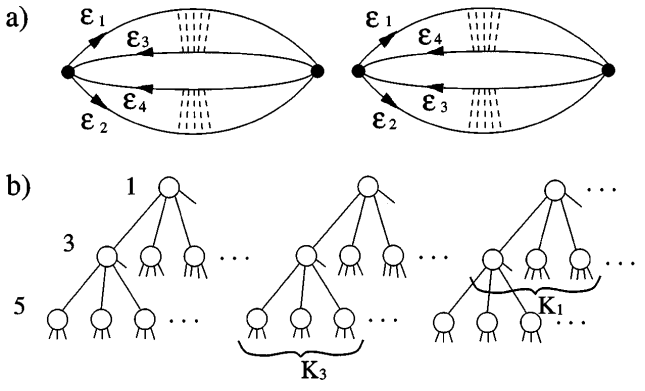

FIG. 1. (a) The diagrams used for evaluating the mean square value of the matrix element in Eq. (4). (b) Schematic representation of the Cayley tree in the Fock space of many-body states. Different generations are shown.

For higher-order generations the DOS rapidly increases: for the $(2n+1)$st generation (where $n < n_{\max} \approx \sqrt{\epsilon/\Delta}$) it goes as $(\epsilon/\Delta)^{2n}/(2n)!$. However, we should focus only on those states of generation $(2n+1)$ which are *directly accessible* from a *given* state of generation $(2n-1)$. The density of such states is much smaller, and is given by $\nu_{2n+1} = \nu_3/n$. We note that from a state in the generation $(2n-1)$ it is also possible to hop to some states of the same generation, and to some states of the previous generation $(2n-3)$. Respectively, DOS associated with these processes is $\sqrt{\nu_3/\Delta}$ and $n(n-1)(2n-3)/\Delta$. For $n \ll n_{\max}$ the number of such hopping processes is parametrically smaller than the number of states in the next generation accessible by hopping. We thus obtain a picture which is quite close to that of a Cayley tree: each "site" of the $(2n-1)$st generation branches out to K_n sites of the next generation. (The number of couplings to the sites of the same or of the previous generations is much smaller, and thus can be ignored [13].) The branching number is given by integrating the effective DOS over the energy interval E_c where the hopping parameter V is energy independent. We obtain the branching number

$$K_n \approx g^3/6n\,. \tag{6}$$

Decreasing K_n with increasing n makes the network effectively finite. To simplify the discussion, below we consider an *infinite Cayley tree* with *constant branching number* $K = K_1 = g^3/6$. Taking into account the finite size of the tree and n dependence of the branching number will be discussed elsewhere.

The model we are interested in was solved by Abou-Chacra, Anderson, and Thouless [14]. They considered localization on a Cayley tree with the on-site energies from a uniform distribution in the interval $[-W, W]$, and constant hopping amplitude V. By studying fix points of the mapping of self-energies computed recursively using the hierarchy of the Cayley tree, it was found that delocalization occurs at

$$Z_c \simeq K \ln K, \quad Z \equiv W/V\,, \tag{7}$$

where K is the tree branching number.

In our case $K \gg 1$, and thus it is important to understand the origin of $\ln K$ which distinguishes Eq. (7) from the original Anderson estimation, $Z_c \simeq K$ [15]. Below we rederive Eq. (7) in a way that displays the structure of states close to the transition and, in addition, clarifies the origin of $\ln K$. For that, we consider the statistics of resonances appearing due to hopping between the tree sites. Starting at a site of the first generation having energy ϵ, the amplitude of hopping to a given site of the $(2n+1)$st generation, in lowest order in V, is given by

$$A_n = \prod_{i=1}^{n} \frac{V}{\epsilon - \epsilon_i}\,. \tag{8}$$

To find the distribution of A_n, we write $\ln(Z^n|A_n|) = \sum_{i=1}^{n} y_i$, where $y_i = \ln(W/|\epsilon - \epsilon_i|)$. Assuming that the relevant on-site energies are uniformly distributed in the interval $-W \le \epsilon - \epsilon_i \le W$, the probability distribution of y_i is $\tilde{P}(y_i) = \exp(-y_i)$ where $0 \le y_i < \infty$. Fourier transforming $\tilde{P}$, taking the nth power and Fourier transforming back, we obtain the distribution function of $\ln|Z^n A_n|$, and then of $|A_n|$,

$$P(|A_n|) = \frac{[\ln(|A_n|Z^n)]^{n-1}}{Z^n(n-1)!\,|A_n|^2}\,, \quad Z^{-n} < |A_n|\,. \tag{9}$$

The typical value of A_n is of the order of Z^{-n}, but we are interested in the rare resonance events when ϵ_n is very close to ϵ, and thus A_n is of the order of one. Let us consider the probability $p(n, C)$ that $|A_n|$ exceeds given finite C, where $Z^{-n} \ll C < 1$. From Eq. (9) we obtain $p(n, C) = \int_C^1 dA\, P(A)$, where at $C < 1$ the integration converges on the upper limit (which thus can be replaced by infinity),

$$p(n, C) \approx \frac{1}{(n-1)!}\,\frac{1}{C\,\ln(CZ^n)}\,[Z^{-1}\ln(CZ^n)]^n. \tag{10}$$

The probability that *none* of the K^n trajectories connecting a site in the first generation to sites in the generation $(2n+1)$ carries a large amplitude is given by

$$[1 - p(n, C)]^{K^n} \equiv \exp(-f_n)\,, \tag{11}$$

where for $p(n, C) \ll 1$, $f_n \approx K^n p(n, C)$. From Eq. (10) for $n \gg 1$ one obtains

$$f_n \approx \frac{Ke}{\sqrt{2\pi n}\,CZ}\left[\frac{Ke}{Z}\,(\ln Z + \ln C/n)\right]^{n-1}. \tag{12}$$

If f_n increases at large n, then at higher generations one gets $f_n \gg 1$, i.e., strong coupling to generation 1. The localization transition takes place when the expression in the square brackets in Eq. (12) reaches unity, which gives the criterion (7).

It is interesting that in the delocalized phase the states are *not ergodic*, i.e., they are not extended uniformly over the whole tree. A typical state occupies an infinite random subtree, formed by a small fraction of sites of the Cayley tree. At large K, one can distinguish two delocalized regimes with qualitatively different structure of states: (i) $K \ll Z \ll K \ln K$ and (ii) $Z \ll K$.

In the case (i) the resonances are typically formed only between distant generations: the probability e^{-f_1} to form a resonance with the nearest generation is small. The characteristic order n_0 of the generation where the first resonance appears can be estimated from the condition

$$(Ke \ln Z/Z)^{n_0} \simeq Z\sqrt{n_0}/K, \tag{13}$$

which means that $n_0 \to \infty$ as $Z \to Z_c$. Resonances in the nearest generation appear only at $Z \simeq K$.

In the case (ii), for each site there are about K/Z resonances in the nearest generation, which means that the effective branching number of an eigenstate subtree is $\sim K/Z$. This is much less than the branching number K of the whole tree; i.e., typical eigenstates do not overlap. Thus, in this regime the states are extended but still not ergodic. Only when $V \simeq W$, and $Z \simeq 1$, the ergodicity is restored.

One may think of the regime (i) as representing an unusually extended crossover region of the transition at $Z = Z_c$. Nonergodicity of extended states on the Cayley tree leads to a number of consequences. For example, the states which contribute to the local spectral density, evaluated for a particular site, are an infinitesimal fraction of all states on the tree. Also, spectral statistics strongly differ from Wigner-Dyson for all $Z \gg 1$.

To translate these results to our problem, we pick the quasiparticle energy ϵ, and approximate the density of states in the generation $(2n + 1)$ accessible from a state of the generation $(2n - 1)$, by a constant $\nu_3(\epsilon)$. The values of V and K are given by Eqs. (5) and (6), respectively, and $K/W = \nu_3(\epsilon)$. We then find that the transition (7) occurs at the energy

$$\epsilon^{**} \approx (\lambda b_d)^{-1/2} \sqrt{\Delta E_c / \ln g}\,. \tag{14}$$

At energies below ϵ^{**} we get localization in Fock space. In the delocalized phase, the transition between the above regimes (i) and (ii) occurs at the energy

$$\epsilon^{*} = (\lambda b_d)^{-1/2} \sqrt{\Delta E_c}\,. \tag{15}$$

Let us discuss the meaning of the various regimes in terms of tunneling DOS. A mathematical description of a single-particle injection into a dot involves projecting a single-particle state onto exact eigenstates of the system. In the localized phase ($\epsilon < \epsilon^{**}$) different generations are weakly connected, and thus at such energies the exact many-body states are close to Slater determinants constructed out of exact single-particle states. Hence, each single-particle state will have a significant overlap with only one (or few) exact eigenstates, producing a few resolved δ-function peaks in the single-particle DOS.

At $\epsilon > \epsilon^{*}$ all generations are well connected. Because of the huge density of multiparticle states, the states of generation 1 can be thought of as being effectively well coupled to the continuum. This justifies the GR result, Eq. (1), in this energy range [9]. Each single-particle peak associated with generation 1 is replaced by a cluster of a large number of many-particle peaks, altogether forming a Lorentzian envelope. For $\epsilon < E_c$, the width of the envelope is less than Δ, and thus the "quasiparticle states" can be resolved in, e.g., transport measurements [5]. We stress that even in this delocalized regime the many-particle states are not ergodic (since at $\epsilon < E_c$ one has $Z > 1$), and therefore the many-particle spectrum is not truly chaotic in the Wigner-Dyson sense (also, see [16]). This should be contrasted with the single-particle states which are chaotic at all energies.

Finally, for intermediate energies $\epsilon^{**} < \epsilon < \epsilon^{*}$ the peaks in the DOS are non-Lorentzian. The peak widths are much smaller than the GR widths because they are determined by couplings between very distant generations on the Cayley tree [see Eq. (13)]. The probability that a particular generation is represented in a given cluster is small. As a result, the widths of the DOS peaks as well as the shapes of their envelopes will strongly fluctuate from peak to peak.

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