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Interactions and the Anderson transition

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We examine the effects of the electron-electron interaction on the Anderson transition. It is shown that the dimensionality of the system and the range of the interaction are crucial in determining the decay properties of a single-particle citation. For a long-range interaction we find that the appropriate one-electron excitations, when localized, decay via a $(\epsilon - \mu)^{1/d}$ law where $(\epsilon - \mu)$ is the energy above the Fermi energy and d is the dimensionality. At finite temperatures this becomes a $T^{1/d+1}$ law. The single-particle excitations are bound for short-range forces. The conditions for the persistence of the Anderson transition are presented in terms of the nature of the "m-basis" (that in which the Green's function is diagonal) and the convergence of a series for the renormalized self-energy.

I. INTRODUCTION

Since 1958, when the concept of localization was first introduced by Anderson,[1] there has been an increasing amount of interest in the field. Applications of the concepts that have grown out of the problem, such as the mobility edge, have been used to explain the behavior of a variety of systems.[2] This has been made possible by a gradual increase in the understanding of the features of the solution to the problem rather than the existence of a complete solution. In fact, the problem has never been solved in detail.

The most prominent apparent success of the application of the theory of localization is in explaining the low-temperature conductivity of certain materials; most notable of these is the inversion layer. Variable range hopping has been observed at low temperatures with an activated conductivity at higher temperatures.[2-4] This is in accordance with the one-electron localization picture. The same sort of behavior has been observed in three dimensions.[2,5,6] On the other hand, the results of measurements of the Hall effect[7] are not well understood. The general state of the art has been reviewed by Mott *et al.*[2] and more recently by Adkins.[8]

Both the successes and failures of the one-electron theory suggest that the problem be examined further. In particular, in this paper, we will look at some of the effects of electron-electron interactions in such a system. Since the localization problem itself is as yet unsolved, one should not expect exact answers for the many-body (Fermi glass) problem. However, by using what we do know about the localization problem, we can improve our understanding of the effects of interactions.

As in the case of the Fermi liquid, the successes of the one-electron picture suggest that we develop a theory of elementary excitations. The possibility that the same physical laws that make this possible for the liquid—the exclusion principle—might result in the same sort of situation in the glass was first suggested in 1970.[9] However, the lack of symmetry in a glass makes the situation quite different. Perhaps the least of the complications is that we do not know the solutions to the noninteracting problem. For even if we did, it is not at all clear that this is the appropriate place to start. We will see that the Fermi glass is by no means "normal" in the sense of the Fermi liquid.

As first argued by Landau, the symmetry of the Fermi liquid leads to a one to one, noncrossing correspondence between the noninteracting and interacting states. Even in a metal, where the symmetry is that of the crystal, this is not necessarily the case. As pointed out by Kohn and Luttinger,[10] the levels may cross and the Fermi surface distorts. The same sort of anomalous diagrams may occur in the glass and lead to mixing so the eigenstates of H_0 with energies below the Fermi energy need not be the correct starting point. Although we will have occasion to return to this point later, we will, for the most part, simply assume that we can self-consistently determine which states to start from and suppose that they are all localized.

Given this we can then see[11] that the lack of momentum conservation leads to single-particle decays that may lead to a finite decay rate at the final Fermi energy. Aside from apparently destroying the quasiparticle picture it would seem that the system can never support a localized perturbation. Since Anderson insulators do seem to occur in nature, this cannot be the final story. In Ref. 11, it was shown how one may eliminate

the single-particle decays by simultaneously diagonalizing the noninteracting Hamiltonian and the irreducible self-energy at the Fermi energy. We will refer to this basis as the "m basis." These states are long lived near the Fermi energy. Whether the system is insulating or not depends, among other things, on whether they are localized or extended.

All of this assumes, as in previous work,[11,12] that these excitations do, in fact, decay. In this paper we will investigate the question more carefully. It will be shown, in perturbation theory, that the single-particle terms vanish, and that the many-particle decay modes that remain are only open for long-range forces. For short-range forces, there is no decay as long as the appropriate states are localized. When the forces are long range one must calculate the rate. The phase space restrictions alone will give an $(\epsilon - \mu)^2$ or T^2 contribution. However, the matrix element will give an additional contribution that will lead to a $T^{1/4}$ law. If we think of the Fermi glass as an energy reservoir for a localized excitation it is not surprising that a $T^{1/4}$ law will be obtained just as with phonon-assisted hopping. The characteristic temperature (T_0 in $\sigma = \sigma_0 e^{-(T_0/T)^{1/4}}$) should be similar in the two cases since it should depend on the decay paths for the excitation. These depend primarily on the nature of the noninteracting system although differences may occur due to the many electrons on the path. The preexponent σ_0 should differ since it is a measure of the rate at which energy can be exchanged with the reservoir and may differ for electron-phonon and electron-electron coupling.

There is another, more physically obvious, problem that is hidden, or concisely included, depending on your point of view, by the m basis. It is the role of spin. The noninteracting system has all of its spins paired and has a Pauli susceptibility. The interacting system will have a large intrastate repulsion that leads to unpaired spins and a Curie susceptibility. When the intrastate repulsion is larger than the bandwidth we may merely ignore spin and allow only one electron per state. When this is not the case we must include the effects such as the screening of deep traps by the first electron and the resulting question of delocalization in the upper Hubbard band. While, in principle, this is contained in the m-basis approach, in practice the effective Hamiltonian must be calculated in perturbation theory. Although there should be no divergences in going from a paired to an unpaired spin system (ignoring the possibility of a metal insulator or magnetic transition), a few terms will not suffice. In the spirit of the Landau theory, one expects the form of the physical quantity to be unchanged except for certain renormalizations, the Landau parameters. This is not the case here, and we would at least expect to be forced to sum an entire class of terms to obtain the correct behavior. So while the m basis is a clean formulation and allows one to make intuitive statements, a physical understanding of the problem is essential.

The outline of this paper will be as follows: The next section will be devoted to a brief review of some of the essential features of the localization problem. In Sec. III we will define the renormalized self-energy [$S_\alpha(\omega)$] used in Ref. 11, and discuss the problems associated in dealing with nontranslationally invariant systems. In Sec. IV we will examine the case of spinless fermions. Such a description is relevant for systems where the intrastate repulsion is larger than the bandwidth and there is a nonintegral number of electrons per site. The doubly occupied states are irrelevant at low temperatures and the particles obey a double exclusion principle. With this assumption we analyze the simplest contributions to the imaginary part of $S_\alpha(\omega)$ and show that the single-particle decay modes are closed and, at most, contribute poles to the spectrum. It is then possible, ignoring crossing, to continue with the eigenstates of H_0. We will do so to make the investigation of range more physical. If the interparticle force is short range, we will see that the many-particle decays are of a similar nature and, therefore, that the spectrum consists of bound states. The "reservoir" cannot effectively exchange energy with the excitation and the low-temperature conductivity should then be due to phonons.

For long-range interactions the many-particle terms form a continuum which makes the single-particle modes irrelevant. In the latter case, one must estimate the decay rate of localized excitations and examine their effect on the conductivity. We will show that the decay rate is small near the Fermi energy, which leads to quasiparticles, and that a $T^{1/4}$ law for the conductivity is obtained. This will be done in Sec. V.

In Sec. VI we will examine the role of spin for finite ($\lesssim$ bandwidth) repulsion. It will be seen that, if the system is still insulating, the form of our previous results is still valid.

As in the localization problem, the overall validity of our conclusions is based on the convergence of the series for $S_\alpha(\omega)$. We will see, in Sec. VII, that this divergence, for some frequency, may renormalize the activation temperature. While in principle this is similar to the noninteracting problem, we will not attempt to

analyze it since the single-particle mobility edge problem has yet to be solved. We will then return to the m basis where we can make plausible the existence of Anderson insulators.

II. THE LOCALIZATION PROBLEM

In this section we will review some of the essential results of the localization problem. We will use these results as a basis for an investigation of the Fermi glass.

One of the most important features of a disordered system is the existence of a mobility edge[13,14] (see Fig. 1). This is the energy which divides the metallic from the insulating states. When the Fermi energy lies in the former region, the system has a finite dc conductivity at zero temperature. In the latter region this quantity would vanish.

The difference in the nature of the states themselves is that they are either extended or localized. Extended states are basically multiply scattered plane waves that occupy a finite fraction of the material. In this region the self-energy is such that the Green's function has a cut line on the real axis; the states are continuumlike and conduct. In the insulating region the states are localized. They are of finite extent and correspond to a region of poles (bound states) on the real axis. We see that these states cannot coexist in the same region. The bound states would simply decay into the continuum.

We expect to find localized states in the band tails, or pseudogaps (regions of low density of states) of samples. States in the band edges rely heavily on coherent scattering to achieve energies that are much different from the original atomic energy levels. We expect these to be the first to lose their ability to tunnel resonantly in the presence of disorder. Therefore these are the first to become localized. The same conclusion is reached by considering the randomness first. Then the density of states is the probability distribution of site energies. States in the tail find fewer states of the correct energy to tunnel to. Consequently, they are more likely to remain localized than those in the center of the distribution.

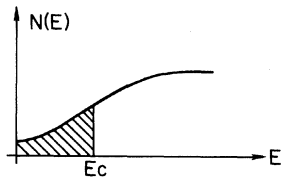

FIG. 1. The density of states as a function of energy. The shaded region represents localized states and Ec is the position of the mobility edge.

We will label the eigenstates of the localization problem by the symbol $|\alpha\rangle$. Because of the lack of symmetry we know that α will consist of the particle's energy ϵ_α^0 and spin. Further, we do not expect the states to be degenerate, except for accidental occurrences of vanishing probability.

It will prove necessary to assume a form for the localized wave functions. We shall write

$$\langle x|\alpha\rangle = \psi_\alpha(x) = \phi_\alpha(\bar{x}) e^{-\lambda_\alpha|\bar{x}-\bar{r}_\alpha|} . \tag{2.1}$$

The function $\phi_\alpha(\bar{x})$ is an oscillatory, incoherent function which may be chosen to be real. The envelope is characterized by the localization length λ_α^{-1}. This quantity is expected to be a regular function of the energy; λ_α should approach zero near the mobility edge and become larger as the energy is moved to the center of the localized regime. The envelope is centered at the point $\bar{r}_\alpha$.

As the Fermi energy moves from below to above the mobility edge, the conductivity at zero temperature changes from zero to some finite value. This is the Anderson (metal-insulator) transition. What is more important from an experimental point of view is the low-temperature behavior of the conductivity. Suppose the Fermi energy μ is below the mobility edge E_c. At low temperatures ($|\mu - E_c| \gg T$) there are very few electrons above E_c. The dominant contribution to the conductivity is due to phonon-induced hopping. The conductivity is proportional to the rate at which an electron added to the system leaves the localized state it was created in. This rate should be proportional to the overlap of the two states involved in the transition ($\sim e^{-\lambda|r_\alpha - r_{\alpha'}|}$) and the probability of there being a phonon of the correct energy ($e^{-\beta(\epsilon_\alpha^0 - \epsilon_{\alpha'}^0)}$). It is statistically unlikely that two states of similar energies will be close together. On the other hand, the probability, at low T, of having a phonon of an energy high enough to scatter the electron to a neighboring site is small. As a consequence there will be an optimal, temperature-dependent distance to hop. This leads to the phenomenon of variable-range hopping.[15,16] The characteristic form of the conductivity is

$$\sigma = \sigma_{ph} e^{-(T_0/T)^{1/d+1}} , \tag{2.2}$$

where d is the dimensionality. (It has been argued that for $d=1$ a "T" law is obtained.[17]) This behavior has been observed and attributed to localized wave functions in a variety of two- and three-dimensional samples.

At higher temperatures, excitation to the mobility edge will become important. Conduction

will be dominated by the population of mobile states. The conductivity should then have the form[6]

$$\sigma = \sigma_{E_c} e^{-|\mu - E_c|/T} . \tag{2.3}$$

This behavior has also been observed. It is also possible to obtain this sort of behavior in a system without a mobility edge or before activation takes place. This would be the Miller-Abrahams[18] regime. In this case $|\mu - E_c|$ should be replaced by a nearest-neighbor energy difference which should be of the order of the bandwidth. The preexponent will also differ.

As mentioned previously, several experiments have shown an anomalous Hall effect: the Hall constant corresponds to the participation of all the carriers and the Hall mobility is activated. It is often argued that in a normal Hall effect, predicted by the single-electron mobility-edge model, only those carriers not trapped below E_c will contribute to the Hall constant (the predicted Hall effect due to hopping[19] has not been observed) and that the Hall mobility should be essentially temperature independent.

It is not clear to us that a simple free-electron mobility-edge model is the correct way to apply the localization picture. We will briefly speculate on this later. For the most part, however, we will concern ourselves with the low-temperature-hopping regime.

III. THE FERMI GLASS AND OTHER NONLIQUIDS

In this section we will prepare to examine some of the single-particle properties of the Fermi glass. While the discussion is in terms of a particular model, it is actually quite general and can be applied to any nontranslationally invariant system where the interactions are treated as a perturbation.

The Hamiltonian will be written in two parts,

$$H = H_0 + U , \tag{3.1}$$

where H_0 is the noninteracting Hamiltonian whose eigenstates $|\alpha\rangle$ are assumed to have the characteristics discussed in the previous section. The term U represents the interaction between particles. In the basis states of H_0 we may write

$$H = \int_\alpha c_\alpha^\dagger c_\alpha \epsilon_\alpha^0 + \tfrac{1}{2} \int_{\substack{\alpha_1 \neq \alpha_2 \\ \alpha_3 \neq \alpha_4}} c_{\alpha_1}^\dagger c_{\alpha_2}^\dagger U_{\alpha_1\alpha_2\alpha_3\alpha_4} c_{\alpha_4} c_{\alpha_3} , \tag{3.2}$$

where

$$U_{\alpha_1\alpha_2\alpha_3\alpha_4} = \int d\bar{x}_1 \int d\bar{x}_2 \psi_{\alpha_1}(\bar{x}_1) \psi_{\alpha_3}(\bar{x}_1) \times V(\bar{x}_1 - \bar{x}_2) \psi_{\alpha_2}(\bar{x}_2) \psi_{\alpha_4}(\bar{x}_2) , \tag{3.3}$$

and $V(\bar{x}_1 - \bar{x}_2)$ is some translationally invariant interaction between the particles.

In order to investigate the single-particle properties of the glass we use the Green's function which is defined by

$$iG_{\alpha\alpha'}(t) = \langle \Psi_0 | Tc_\alpha(t) c_{\alpha'}^\dagger(0) | \Psi_0 \rangle . \tag{3.4}$$

The decay rate of a localized excitation is determined by the diagonal part of the Green's function. The Fourier transform of this quantity satisfies the matrix equation

$$G_{\alpha\alpha}(\omega) = G_{\alpha\alpha}^0(\omega) + \int_{\alpha_1} G_{\alpha\alpha}^0(\omega) \Sigma_{\alpha\alpha_1}(\omega) G_{\alpha_1\alpha}(\omega) , \tag{3.4'}$$

where Σ is the (irreducible) self-energy and G is the noninteracting Green's function which is given by

$$G_{\alpha\alpha'}^0(\omega) = \delta_{\alpha\alpha'} \lim_{\eta \to 0^+} [\omega - \epsilon_\alpha^0 - i \operatorname{sgn}(\mu - \epsilon_\alpha^0)\eta]^{-1} . \tag{3.5}$$

In the Fermi liquid, translational invariance allows Eq. (3.4) to be put in an algebraic form. Together with the relation analogous to (3.5), we can solve for the Green's function in terms of the self-energy. This allows us, in principle (given Σ) to find the poles and the decay rate. This is not the case in the Fermi glass, nor in other nonliquids where G and Σ are not diagonal. However, we can show that $G_{\alpha\alpha}(\omega)$ may be written in the form

$$G_{\alpha\alpha}(\omega) = [\omega - \epsilon_\alpha^0 - S_\alpha(\omega)]^{-1} , \tag{3.6}$$

where $S_\alpha(\omega)$, the "renormalized" self-energy is given by

$$S_\alpha(\omega) = \Sigma_{\alpha\alpha}(\omega) + \int_{\alpha_1 \neq \alpha} \Sigma_{\alpha\alpha_1}(\omega) G_{\alpha_1\alpha_1}^0(\omega) \Sigma_{\alpha_1\alpha}(\omega) + \cdots . \tag{3.7}$$

The proof of this statement is an exercise in algebra. However, it does have a simple interpretation.

Suppose we ignore the terms in the sum in Eq. (3.4') that have $\alpha_1 \neq \alpha$. Then the Green's function would have the form in (3.6) with $\Sigma_{\alpha\alpha}(\omega)$ replacing $S_\alpha(\omega)$. Since $\Sigma_{\alpha\alpha}(\omega)$ is irreducible it has no single internal propagator lines. Therefore, the imaginary part represents no transitions from the state $c_\alpha^\dagger|\Psi_0\rangle$ to some other state $c_{\alpha'}^\dagger|\Psi_0\rangle$. In the Fermi liquid such a process would not conserve momentum, and if the two states are identical the process does not represent a decay. This is why $\Sigma_k(\omega)$ appears in the Green's function for the Fermi liquid. In the Fermi glass there is no

such restraint. These decays must be represented. To $\Sigma_{\alpha\alpha}(\omega)$ we must add all possible single-particle decays except for those states with $\alpha_j = \alpha$ since these are not transitions (see Fig. 2). This is the meaning of Eq. (3.7).

The imaginary part of $S_\alpha(\omega)$, which determines the lifetime of the state $c_\alpha^\dagger|\Psi_0\rangle$, has two kinds of contributions. One is from the many-particle state associated with Σ; the other is that which comes from the single-particle states or the single propagators. The former give contributions that are small near the Fermi energy. The reason is that the same phase-space restrictions that are the cause of quasiparticle behavior in a liquid are operative in the Fermi glass. More technically, this behavior is due to any of the $2n+1 (n>0)$ internal lines that may be found in Σ. Therefore, it is true that

$$\lim_{\omega\to\mu} \mathrm{Im}\Sigma_{\alpha\alpha'}(\omega) = 0 . \tag{3.8}$$

This conclusion is not restricted to the Fermi glass. It is, however, dependent on perturbation theory.

The single-particle terms do not have this property. In general, especially in a system without essential disorder, there is no reason why there should not be a finite contribution to the imaginary part of $S_\alpha(\omega)$ as ω approaches μ. This has two immediate consequences. The first is that the bare particle created by $c_\alpha^\dagger$ is short lived. It is unlikely that it can be filtered to a quasiparticle state. While this is generally true for nonliquids, there is another consequence for the Fermi glass. If we create a localized state at μ, it will decay quickly. The ground state is not capable of sustaining a localized excitation and we see that the insulating behavior is destroyed. This is, in part, the reason that the m basis was previously introduced.[11]

In the Fermi glass, however, we must consider the other problem mentioned in the Introduction. This comes down to asking whether $\mathrm{Im}S_\alpha(\omega)$ is finite and continuous, or vanishes except at a discrete set of points. That is, we must decide whether the excited states are bound or continuum-like. We shall see, in the next section, that this depends on the range of the interaction and the dimensionality of the system.

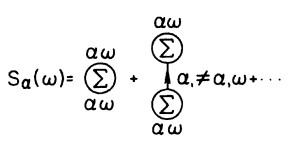

FIG. 2. A graphical representation of $S\alpha(\omega)$.

Before continuing, we should point out that $S_\alpha(\omega)$ has many of the same analytic properties as the usual $\Sigma_k(\omega)$. In addition it is quite simple to extend the concepts in this paper to finite temperature. One should also remember the introductory remarks on normal systems. In principle, we should be using some self-consistently obtained states and Fermi energy.

IV. CONTINUUM VS BOUND STATES

In this section we wish to see whether a localized excitation in the Fermi glass can exchange energy with the other electrons and diffuse away. To do this we must analyze the imaginary part of the renormalized self-energy. We will assume, unless stated otherwise, that μ lies in the region of localized states and that the relevant energy, as specified by ω, is such that $\omega < E_c$.

There are basically two types of terms in the imaginary part of $S_\alpha(\omega)$: those representing single-particle decay modes and those representing many-particle modes. (There are also terms that involve interference between the two. These will also have phase-space restrictions, and we will classify them in the latter category.) The lowest-order (direct) diagrams of both are shown in Fig. 3. They correspond to the application of Fermi's "golden rule" to the simplest decays of the state $c_\alpha^\dagger|\Psi_0\rangle$.

Let us first examine the structure of the process described by Fig. 3(a). The range of the interaction is "horizontal." That is, the states $|\alpha\rangle$ and $|\alpha_1\rangle$ are functions of the same variable in the expression for the matrix element. Since we are interested in $\omega \gtrsim \mu$ (the case $\omega \lesssim \mu$ is similar), the state $|\alpha_1\rangle$ will be localized. In order to obtain a result greater than some number, we must require that $|\alpha_1\rangle$ be localized in the neighborhood of $|\alpha\rangle$. This restricts us to a finite number of states $|\alpha_1\rangle$. When we take the imaginary part of this diagram we further require that $\epsilon_{\alpha_1}^0$ be equal to ω. The probability of this condition being satisfied over the finite number of points, corresponding to the contributing states $|\alpha_1\rangle$, is zero. Therefore, on the real axis, this contribution can, at most, look like the sum of delta functions, or, in the complex plane we would obtain poles on the real axis.

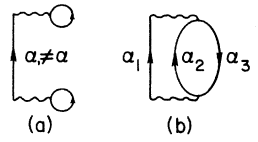

FIG. 3. The lowest-order (direct) contributions to $S\alpha(\omega)$.

The structure of the second diagram in Fig. 3 is somewhat different. Taking the imaginary part restricts us to a plane of area $\sim(\omega-\mu)^2$ in the space $(\epsilon^0_{\alpha_1},\epsilon^0_{\alpha_2},\epsilon^0_{\alpha_3})$ (see Fig. 4). Again $|\alpha_1\rangle$ must be near $|\alpha\rangle$. This defines a finite set of planes, which intersect the $\epsilon^0_{\alpha_1}$ axis and are parallel to the $(\epsilon^0_{\alpha_2},\epsilon^0_{\alpha_3})$ plane. These planes define a finite number of lines in the original triangle. We should also require that $|\alpha_2\rangle$ be near $|\alpha_3\rangle$. However, $|\alpha_3\rangle$ may be anywhere, and we obtain a dense set of points on the $(\epsilon^0_{\alpha_2},\epsilon^0_{\alpha_3})$ plane. The probability of at least one of these intersecting the lines in the phase space plane is finite.

The question now becomes whether or not we should require the pair $|\alpha\rangle, |\alpha_1\rangle$ to be near the pair $|\alpha_2\rangle, |\alpha_3\rangle$. Clearly this depends on the range of the interaction. By range we do not mean as a function of $|\vec{x}-\vec{x}'|$, but of R, the distance between the pairs. If the interaction is long range we need make no further requirements of the states involved. For arbitrary ω we obtain a finite contribution from this diagram. (This is true until $\omega\sim 1/\Omega$, then we may have no available states that contribute. That is, the plane in Fig. 4 may not be intersected at all.) For short-range forces we must require that the *pairs* be close together. This further requirement leads to a vanishing of the diagram except for special values of the frequency in a manner similar to the single-particle terms. An analysis of higher-order terms leads to the same sort of conclusions for the two cases.

Before we continue and analyze the consequences of these possibilities, let us consider the problem of range. Unfortunately for our theory our ability to answer this question is limited by our understanding of the localization problem. We see that we must be able to "hop" from one pair to the other via the range of U. In Anderson's theory,[1] if the range falls off slower than, or as slow as, R^{-3}, we are guaranteed that this can be done. Now the transition shown in Fig. 3 corresponds to an "oscillation" of charge. Therefore, the range, in terms of R, is that of interacting dipoles on R^{-3}. This puts us on the borderline, though on the continuum side. The R^{-3} is not particularly well understood.

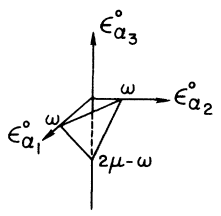

FIG. 4. The phase-space restrictions on the decay represented by Fig. 3(b). The axes are the unperturbed energies of the states α_1, α_2, and α_3.

One may think of effects which tend to push the range either way. For example, there is spin diffusion. This will tend to broaden the states, and effectively push us towards continuum behavior [finite $\mathrm{Im}S_\alpha(\omega)$ for arbitrary ω]. This is similar to the case in phonon-induced hopping. The coupling to the phonons, which are a continuum, allows one to absorb any amount of energy, and if the phonons are extended the decay may always proceed. On the other hand, one may argue that if we have continuum states they will screen and reduce the range of the interaction. However, we will then have bound states which will not screen. Clearly, the problem must be solved self-consistently or in terms of summation over the polarization parts of the interaction line. We should also point out that the contribution from the term with $\alpha_1=\alpha$ involves the generation of the dipole pair in the presence of a monopole and goes like R^{-2}. It would seem, though that this involves dressing of the bare particles and should not play an essential role (although what is not essential in the borderline case is, perhaps, not so clear).

We will not attempt to solve the problem of range in this paper. Rather, we will suppose that the interaction may be either short or long range and investigate both possibilities. It should be noted, however, that if a sample is really two dimensional, and if the electric field may extend outside of it, that R^{-3} is not borderline but short range. In general, one would expect R^{-4} to be the crossover.

Let us first consider the consequences of short-range forces. In this case all the terms are discrete. If the perturbation theory is valid, we expect the spectrum to consist of bound states up to the mobility edge. At these energies the arguments made above no longer apply since the states $|\alpha_1\rangle$ may be extended. The spectrum for excitation energies above E_c should then consist of continuum states. The lower energy states will be bound and the low-temperature conductivity should be due to phonons. Since nearby states will tend to have large energy differences, and those with small energy differences are far away, the excitation should induce a polarization that is largest at some distance. This annular cloud, which is thus weaker than might otherwise be expected, will dress the bare particle and tend to increase the energy barriers involved in the hopping. At higher temperatures contributions to the conductivity from the continuum states should give an activated form.

If the perturbation theory for $S_\alpha(\omega)$ diverges

for some frequency, we expect the mobility edge to be moved. The question of where this happens is, at this time, not of any great importance since no rigorous method of calculating E_c exists for noninteracting systems. We should point out, however, that since the nature of the basis of expansion changes at E_c, it is possible that the edge may be normalized up or down. It is, however, possible that the series may diverge right down to μ. In this case, the system may not remain insulating. We will deal with this in Sec. VII.

In the case of long-range interactions we have continuum behavior over the entire spectrum. The single-particle terms are unimportant insofar as $\mathrm{Im}S_\alpha(\omega)$ is concerned. Then the phase-space restrictions, and other effects (see Sec. V) give quasiparticle behavior. We can create a stable localized excitation at μ which implies that the ground state is nonconducting. The form of the conductivity must be determined, and we will do so in the next section. At higher temperatures excited states of a differing nature (they have real transitions above E_c) come into play and the conductivity should appear activated. Of course, the series may diverge and change E_c or even suppress the transition. We will deal with this possibility in Sec. VII.

V. QUASIPARTICLES AND THE $T^{1/4}$ LAW

In the case of long-range interactions we will have continuum behavior. It is then permissible to choose a model with which to calculate quantities. We will use the same model as used in the theory of variable-range hopping. In this latter case, the use of the model is justified because, as we have mentioned previously, the electrons are coupled to the extended phonon continuum.

Let us calculate the contribution to the imaginary part $S_\alpha(\omega)$ due to the diagram shown in Fig. 2(b). We will assume that ω is close to μ and in a region where the series converges. The single-particle decays should then be unimportant and higher-order terms, if of the same form, should only renormalize quantities, not change the essential form of the result.

Say ω is greater than μ. The contribution from Fig. 3(b) is

$$\mathrm{Im}\Sigma_{\alpha\alpha}(\omega)=2\pi\int_{\alpha_1\alpha_2\alpha_3}(U_{\alpha\alpha_3\alpha_1\alpha_2})^2\theta(\epsilon^0_{\alpha_1}-\mu)\theta(\epsilon^0_{\alpha_2}-\mu)\theta(\mu-\epsilon^0_{\alpha_3})\delta(\omega+\epsilon^0_{\alpha_3}-\epsilon^0_{\alpha_1}-\epsilon^0_{\alpha})$$
$$=2\pi\Omega^3\int_\mu^\omega d\epsilon^0_{\alpha_1}\int_\mu^{\omega+\mu-\epsilon^0_{\alpha_1}}d\epsilon^0_{\alpha_2}N(\epsilon^0_{\alpha_1})N(\epsilon^0_{\alpha_2})N(\epsilon^0_{\alpha_1}+\epsilon^0_{\alpha_2}-\omega)(U_{\epsilon^0_\alpha\epsilon^0_{\alpha_1}}+\epsilon^0_{\alpha_2}-\omega\epsilon^0_{\alpha_1}\epsilon^0_{\alpha_2})^2\,, \tag{5.1}$$

where N is the density of states/energy volume and Ω the volume. The matrix element is given by

$$\int d\bar{x}\int d\bar{x}'\psi_{\epsilon^0_\alpha}(\bar{x})\psi_{\epsilon^0_{\alpha_1}}(\bar{x})V(\bar{x}-\bar{y})\psi_{\epsilon^0_{\alpha_2}}(\bar{x}')\psi_{\epsilon^0_{\alpha_1}+\epsilon^0_{\alpha_2}-\omega}(\bar{x}')\,.$$

Using the results of Sec. II, and realizing that the low energies involved will keep the states relatively far apart, we write

$$\phi_{\epsilon^0_\alpha}(\bar{x})\phi_{\epsilon^0_{\alpha_1}}(\bar{x})=g_{\alpha\alpha_1}(\bar{x})e^{-\lambda|\bar{r}_{\alpha_1}-\bar{r}_\alpha|}\,. \tag{5.2}$$

The quantity λ may be evaluated at μ; $g_{\alpha\alpha_1}(\bar{x})$ is a function that is localized between the sites $\bar{r}_\alpha$ and $\bar{r}_{\alpha_1}$. Following Mott,[20] we write

$$|\bar{r}_\alpha-\bar{r}_{\alpha_1}|\approx\left(\frac{3}{4\pi N_\mu|\epsilon^0_\alpha-\epsilon^0_{\alpha_1}|}\right)^{1/3}. \tag{5.3}$$

A similar argument applies for the other pair of wave functions.

The exponentials do not depend on $\bar{x}$ or $\bar{x}'$ and may be removed from the matrix element. The remainder, the reduced matrix element, is called $\hat{U}$. Since all the energies are near μ, we may, having extracted the singular behavior, evaluate this quantity along with the other slowly varying quantities at μ. [The question of whether or not $N(\epsilon)$ is slowly varying has been the subject of some controversy.[21,22,23] We will assume, here, that it is slowly varying. This is contingent on the regularity of $\mathrm{Re}S_\alpha(\omega)$.] We have then

$$\mathrm{Im}\Sigma_{\alpha\alpha}(\omega)\approx 2\pi\hat{U}^2\Omega^3N_\mu^3\int_\mu^\omega d\epsilon^0_{\alpha_1}\int_\mu^{\omega+\mu-\epsilon^0_{\alpha_1}}d\epsilon^0_{\alpha_1}\exp\left[-\left(\frac{\hat{a}}{\epsilon^0_\alpha-\epsilon^0_{\alpha_1}}\right)^{1/3}-\left(\frac{\hat{a}}{\omega-\epsilon^0_{\alpha_1}}\right)^{1/3}\right], \tag{5.4}$$

where

$$\hat{a}=\frac{3\lambda^3 2^3}{4\pi N_\mu}. \tag{5.5}$$

As $\omega\to\mu$ the area of the integrand goes like $(\omega-\mu)^2$ but the second exponential goes to zero. We can (over) estimate the integral by

$$2\pi\hat{U}^2\Omega^3N_\mu^3(\omega-\mu)^2\exp\left[-\left(\frac{\hat{a}}{\epsilon^0_\alpha-\mu}\right)^{1/3}-\left(\frac{\hat{a}}{\omega-\mu}\right)^{1/3}\right].$$

There are two parts. The first is the phase-space contribution $(\omega-\mu)^2$. The second is due to the localized structure of the material. The real part of Σ, if sufficiently regular, will renormalize ϵ_α^0 to E'_α. We can then estimate the decay rate to be

$$\frac{1}{\tau'_{E_\alpha}} \sim (E'_\alpha-\mu)^2 \exp\left[-2\left(\frac{\hat{a}}{E'_\alpha-\mu}\right)^{1/3}\right], \tag{5.6}$$

which gives us quasiparticle behavior. (A better estimate is obtained using the incomplete gamma function. The quadratic nature of the phase space is somewhat modified.) The higher-order contributions are also exponential. Let us anticipate this and restrict our attention to two-particle (hole), single-hole (particle) states. These terms are proportional to $(\omega-\mu)^2$. It will, in general, take n hops to reach a vertex P from the state P' shown in Fig. 5. The particle-hole pair at P' will be a distance proportional to $(\omega-\mu)^{-1/3}$. Of course we can have $n-1$ small hops but we must then have one large one. Therefore we will always have a product of exponentials leading to a term, for a given path, of the form (5.6). The quantity analogous to $\hat{a}$ depends on the path chosen and, for low enough energies, may be dominated by the optimal path. In general, since all terms propagate exponentially, we expect the series to renormalize the quantities in the lifetime and we obtain

$$\frac{1}{\tau_{E_\alpha}} \sim (E_\alpha-\mu)^2 \exp\left[-\left(\frac{\hat{a}}{E_\alpha-\mu}\right)^{1/3}\right]. \tag{5.7}$$

It is easy to show that in d dimensions $\frac{1}{3}$ becomes $1/d$.

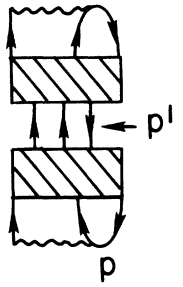

FIG. 5. Cutting this diagram at p^1 results in a higher-order decay to the same final state shown in Fig. 3(b). The same sort of decay rate is obtained.

One may ask whether virtual transitions above the mobility edge will wipe out the exponential behavior; however, these states are limited in the distance they can propagate by the uncertainty principle (since $h \sim 10^{-11}$ meV sec they are very short lived). Therefore, this effect should be unimportant in this sense. These processes represent a renormalization of the energies and wave functions involved and may therefore alter the parameters in the $T^{1/4}$ law. If the wave functions are changed enough to become extended, the series for S_α should diverge when α is localized.

Using this result, one can argue that the contribution to the conductivity is a $T^{1/d+1}$ law. It is perhaps more physically illuminating to do the calculation directly for finite temperature.

Let us use Fermi's golden rule to calculate the decay rate of our localized state at finite temperature. We have

$$\frac{1}{\tau_{\epsilon_\alpha^0}} = \int_{\alpha_1\alpha_2\alpha_3} U^2_{\alpha\alpha_2\alpha_1\alpha_3}\,\delta(\epsilon_\alpha^0+\epsilon_{\alpha_3}^0-\epsilon_{\alpha_2}^0-\epsilon_{\alpha_1}^0) f(\beta(\epsilon_{\alpha_3}^0-\mu)) f(\beta(\mu-\epsilon_{\alpha_1}^0)) f(\beta(\mu-\epsilon_{\alpha_2}^0)), \tag{5.8}$$

where $\beta = T^{-1}$ and $f(x) = 1/(e^x+1)$. In this case the phase space will restrict the energies to lie within T of the Fermi energy. Using the same model as the zero temperature case gives

$$\frac{1}{\tau_{\epsilon_\alpha^0}} = 2\pi\Omega^3 N_\mu^3 \hat{U}^2 \int d\epsilon_{\alpha_3}^0 \int d\epsilon_{\alpha_2}^0 f\left(-\frac{\epsilon_{\alpha_2}^0}{T}\right) f\left(\frac{-\epsilon_\alpha^0-\epsilon_{\alpha_3}^0+\epsilon_{\alpha_2}^0}{T}\right) f\left(\frac{\epsilon_{\alpha_3}^0}{T}\right) \exp\left[-4\left(\frac{a}{\pm\epsilon_{\alpha_3}^0 \mp \epsilon_{\alpha_2}^0}\right)^{1/3}\right], \tag{5.9}$$

where we are measuring all quantities from the Fermi energy and $\epsilon_{\alpha_3}^0 \gtrless \epsilon_{\alpha_2}^0$. We see that the phase causes the integrand to be constrained within T of μ. However, the matrix element is zero when $\epsilon_{\alpha_2}^0 = \epsilon_{\alpha_3}^0$. The result is an integrand with a peak. We locate the peak for $\epsilon_{\alpha_3}^0 \lesseqgtr \epsilon_{\alpha_2}^0$ by calculating the derivatives of the integrand and setting them equal to zero. Since

$$\frac{d}{dx} f(x) = -f(x)f(-x),$$

we have

$$f\left(\frac{\epsilon_{\alpha_2}^0}{T}\right) - f\left(\frac{\epsilon_\alpha^0+\epsilon_{\alpha_3}^0-\epsilon_{\alpha_2}^0}{T}\right) \mp \frac{\frac{4}{3}a^{1/3}T}{(\pm\epsilon_{\alpha_3}^0 \mp \epsilon_{\alpha_2}^0)^{4/3}} = 0, \tag{5.10}$$

$$f\left(\frac{\epsilon_\alpha^0+\epsilon_{\alpha_3}^0-\epsilon_{\alpha_2}^0}{T}\right) - f\left(\frac{-\epsilon_{\alpha_3}^0}{T}\right) \pm \frac{\frac{4}{3}a^{1/3}T}{(\pm\epsilon_{\alpha_3}^0 \mp \epsilon_{\alpha_2}^0)^{4/3}} = 0, \tag{5.11}$$

adding gives

$$\epsilon_2 = -\epsilon_3 .$$

Substituting this yields (take $\epsilon_2 \lesseqgtr 0$)

$$f\left(\frac{\epsilon_2}{T}\right)-f\left(\frac{\epsilon_\alpha^0-2\epsilon_2}{T}\right)\pm\frac{\frac{4}{3}a^{1/3}T}{(\mp 2\epsilon_2)^{4/3}}=0\,. \tag{5.12}$$

For $\epsilon_2>0$ the third term drops down more slowly than the first and the second rises. Therefore, ϵ_2/T will be rather large as T is lowered (ϵ/T is understood to be small. As T goes to zero we pass from the thermal regime into the case discussed previously). The first term can be dropped and the second is about one. For low temperature or well localized states, this is an excellent approximation. A similar argument holds for $\epsilon_2<0$ and we obtain

$$-\epsilon_3=\epsilon_2=\frac{a^{1/4}}{2}\left(\frac{4T}{3}\right)^{3/4}.$$

Since the area of the integrand is about T^2 we obtain (in d dimensions)

$$\frac{1}{\tau}=2\pi\Omega^3N_\mu^3\hat{U}^2T^2\exp\left\{-\left[4\left(\frac{d}{4}\right)^{1/d+1}+\left(\frac{4}{d}\right)^{d/d+1}\right]\left(\frac{a}{T}\right)^{1/d+1}\right\}. \tag{5.13}$$

Thus a localized excitation will decay away by the $T^{1/4}$ law. This rate is related to the diffusion constant which is in turn related to the conductivity. Thus we should expect that our system should have a conductivity obeying a $T^{1/4}$ law as claimed previously. The higher terms in the series should modify the quantities in the law, but not the form of the relation itself. So, for example, while $\hat{a}$ should certainly be inversely proportional to N, the constant of proportionality may vary. The same comments would hold for the phonon case.

When the temperature is high enough, contributions from extended states will be important and the conductivity will be activated. Similarly, if the Fermi energy lies above E_c, we expect $(\omega-\mu)^2$ or T^2 behavior and zero resistivity, rather than zero conductivity, would be our starting point.

VI. THE ROLE OF SPIN

Until now we have been using an analysis that is meant, primarily, for electrons obeying a double exclusion principle. When the intrastate repulsion is small enough, one has to consider the effects of two electrons in the same state. As far as the Anderson transition goes, this means asking two questions: first, is the upper Hubbard band occupied, and second, is it extended or localized? The former question can be answered by examining whether the intrasite repulsion vanishes more or less quickly than $E-E_c$ as E approaches E_c from the localized region. Numerical work[24] indicates that the repulsion stays higher and only the lower Hubbard band remains occupied for a less than half-filled band. In this case an electron is excluded from visiting many (all for a half-filled band) sites and delocalization is harder to achieve. In fact most Anderson insulators seem to be paramagnets.

When two electrons per state are allowed, the amount of disorder seen by the second electron is reduced by the first. Metallic behavior is more likely. Experimentally it seems that only the tail of the upper Hubbard band may be localized.

If we do have an insulator with doubly occupied states we should still expect to see $T^{1/4}$ behavior. A low-energy excitation, be it an electron or phonon, may generate a low-energy particle-hole pair in either of the two bands. By an argument similar to that in Sec. V this will give rise to a variable-range behavior. It is also possible, for example, to remove a particle from the upper Hubbard band and excite it to the lower. For large U this, at first sight, might be expected to cause trouble since the single-particle states involved differ by U. But if we allow the second electron in the state to relax in the field of the first, its energy will rise to near the chemical potential. Then, if it is close to the unoccupied lower Hubbard bound state, the wave functions will mix and the levels will split. Therefore these cannot be the states between which the transition we are investigating takes place and we see that the levels will be far apart and the $T^{1/4}$ law should occur.

It is not necessary, however, that all of these processes involve the same parameters. In particular the mobility edge for the upper and lower Hubbard bands, or, more precisely, for particles and holes, need not be the same.

Freedman and Hertz[12] have obtained a Pauli spin susceptibility for the low-temperature Fermi glass. Their method uses the noninteracting Hamiltonian to represent the simple system. In view of the strong intrasite repulsion and the fact that in the spirit of Fermi liquid theory the interactions should just renormalize quantities not change the essential form of a relation, this result should be viewed with some caution. As we have pointed out, it should take some form of partial summation to correctly describe the interacting system if one starts with the eigenstates of H_0. A better starting point would be

$$H_0 \to H_0'=H_0+\sum_\alpha U_\alpha n_{\alpha\uparrow}n_{\alpha\downarrow}\,, \tag{6.1}$$

so that the repulsion is considered explicitly. The unpaired spins will give a Curie law. It has recently come to our attention that Kamimura[25] has cal-

culated the susceptibility with the assumptions that $U_\alpha = u$ and that the ϵ_α^0 are random variables. He finds that for large u the low-temperature behavior is T^{-1} with a background Pauli contribution that is due to the unpaired spins.

All of this, of course, is only true if the system is nonmagnetic. Presumably, as pointed out by Mott,[26] the interaction between states is antiferromagnetic and either a spin glass or disordered antiferromagnet will result. Until now, the transition has yet to be observed.[26]

VII. THE m BASIS

Until now, our arguments have been based on the leading terms in the perturbation series. We have seen that the decay properties of low-energy excitations depend on the range of the interaction and the dimensionality of the sample. These arguments hold until the energy becomes such that states above the mobility edge become important. At this point the nature of the transport process changes and the conductivity will appear activated.

The variable-range (or nondecay) behavior comes about because the single-particle decay modes are closed. Since the proper self-energy Σ is short ranged (exponential), this is true in perturbation theory. The question is whether the series for S converges. (Actually a renormalized version should be analyzed.) At energies where there is a divergence, the single-particle modes become opened, and the nature of the excited state is different. Thus, the activation temperature may be renormalized. To ask exactly where this occurs for a given system is not profitable at this time since the corresponding one-electron problem remains unsolved. Further, as mentioned earlier, there is the possibility of level crossing. Even if one knew the eigenfunctions of H_0, the crossing may lead to problems in trying to construct, for example, the ground state out of something destined to be an excited state.

However, a more important question is whether or not the system will remain insulating in the presence of interactions. If μ is less than E_c in the noninteracting system and our first term analysis is good, we can create the ground state by adding localized particles at the Fermi energy. We have a nonconducting ground state. This is not the case if the series diverges in the neighborhood of μ. As we have seen in the last section, spin and the intrasite repulsion may play an important role in this question. It is possible to take a different approach which expresses all of these effects in a concise way and gives a criterion for insulating behavior that allows one to understand the existence of localization in interacting systems.

To this end consider Dyson's equation written in the form [one may wish to include (6.1) in H_0]

$$[\omega - H_0 - \Sigma(\omega)]G(\omega) = 1 . \tag{7.1}$$

Suppose that the operator $H_0 + \Sigma(\omega)$ can be diagonalized. Let us call the eigenstates $|m(\omega)\rangle$ and the eigenvalues $\epsilon_{m(\omega)}(\omega)$. Then the Green's function is diagonal and has the form

$$G_{m(\omega)m(\omega)}(\omega) = \frac{1}{\omega - \epsilon_{m(\omega)}(\omega)} . \tag{7.2}$$

However, $\Sigma(\omega)$ is not, in general, a Hermitian operator. Therefore, the eigenvalues $\epsilon_{m(\omega)}(\omega)$ need not be real, nor the $|m(\omega)\rangle$ orthogonal. In fact, there is no guarantee that we can diagonalize $H_0 + \Sigma(\omega)$ at all. (In the Fermi liquid we can do this in the momentum representation.) However, $\Sigma(\mu)$ is Hermitian. In fact it is a real symmetric matrix. This is because the imaginary part of Σ is zero as shown in Sec. IV. In a probabilistic sense (allowing the possibility of discrete behavior) the space over which the arguments of Sec. IV were made goes to zero. Therefore, the imaginary part of Σ vanishes with probability unity at μ.

For $\omega \approx \mu$ we can write $H_0 + \Sigma(\omega)$ as

$$H_0 + \Sigma(\mu) + \delta\Sigma(\omega) + i\,\mathrm{Im}\Sigma(\omega) , \tag{7.3}$$

where $\delta\Sigma$ and $\mathrm{Im}\Sigma$ are both small. Then the transformation to the $|m(\omega)\rangle$ is almost canonical or pseudocanonical. We will return to the case $\omega \neq \mu$ in a moment.

At μ, G can be written

$$G_{m(\mu)m(\mu)}(\mu) = \frac{1}{\mu - \epsilon_{m(\mu)}(\mu)} . \tag{7.4}$$

The pole [which defines a state $|m_0(\mu)\rangle$] is real. If the state $|m_0(\mu)\rangle$ is localized, we can argue, by methods used earlier, that the ground state is nonconducting.

The advantage in this approach is that it is clear, in at least some limit, that there is no conductivity at zero temperature. To obtain $|m(\mu)\rangle$ we must diagonalize a random matrix: $H_0 + \Sigma(\mu)$. The first part of this operator was disordered enough to localize at the states in the neighborhood of the energy we are considering. To this we add $\Sigma(\mu)$, which is also a disordered matrix. It is tempting to say that the effective Hamiltonian is even more disordered than the initial Hamiltonian and the states must remain localized. However, whatever correlations the self-energy has come from the structure of H_0. So the former conclusion is not obvious. On the other hand, in the strongly localized limit, it is clear that this sort of conclusion must hold. Furthermore, for a large intrastate repulsion

there will be double exclusion principle and the particle cannot visit all of the sites, which makes it easier to localize.

For frequencies near μ the form of (7.2) should be approximately valid, and we can obtain the eigenvalues (poles) via Rayleigh-Schrödinger perturbation theory. In particular, we are interested in the imaginary part

$$\mathrm{Im}\epsilon_{m(\omega)}(\omega) = \langle m(\mu)|\,\mathrm{Im}\Sigma(\omega)\,|m(\mu)\rangle + \cdots . \quad (7.5)$$

As we have argued previously, $\mathrm{Im}\Sigma_{\alpha\alpha'}(\omega)$ must approach zero as ω approaches μ. This is due to phase space considerations alone, and does not depend on the system being considered (providing of course, that the Fermi surface exists; a superconductor is not what we have in mind). We thus obtain quasiparticle behavior that is at least of the $(\omega-\mu)^2$ or the T^2 sort.

If we return to the Fermi glass, it is not difficult to show for long-range forces that all the matrix elements of $\mathrm{Im}\Sigma$ have an exponential behavior when $\omega < E_c$. If the $|m(\mu)\rangle$ is localized we can then imagine expanding it in the basis. In this case terminating the series should produce arbitrarily good results and $\mathrm{Im}\Sigma_{m(\omega)}(\omega)$ should behave in the same anomalous way as before. Actually we expect the expansion to be dominated by a single term. The state $|m(\mu)\rangle$ should strongly overlap only a few nearby states, only one of which will have an energy close to $\epsilon(m(\mu))$. The rest will have large energy denominators and be less important. Similarly other states with ϵ_α^0 close to $\epsilon(m(\mu))$ will tend to be far away and overlap poorly. Therefore, we reproduce our previous results. When the force is short ranged, the matrix elements of $\mathrm{Im}\Sigma(\omega)$ vanish and so does $\mathrm{Im}\epsilon_{m(\omega)}(\omega)$.

One may ask about the effects of extended states. First, we note that the matrix elements of $\Sigma(\omega)$ between such states still have the exponential behavior due to the particle-hole pair. Second, this contribution to $\mathrm{Im}\epsilon(m(\omega))$ is not that important since they are suppressed by $(\mu - E_c)^{-1}$. That these terms work out this way should not be surprising, since it is ω, not the indices on Σ, that set the appropriate energy scale. Of course, as μ and ω approach E_c, these become more and more relevant and ultimately may lead to metallic behavior. In this case $|m(\mu)\rangle$ becomes extended and we expect to find the normal phase-space behavior.

VIII. DISCUSSION

We have seen that the nature of the system depends on the range of the forces between the particles. If the interaction is short range, the excited states of the system will be nonconducting. This should be true up to energies which begin to include transitions that involve real extended states. The analysis is independent on perturbation theory. The convergence of the series as a function of frequency may renormalize the mobility edge. Conduction at low temperature will be due to phonon induced hopping of localized quasiparticles and their polarization clouds. This should give a $T^{1/4}$ law at low temperatures with activated behavior at higher temperatures.

The situation is somewhat different for long-range forces. The low-energy excitations will decay anomalously and give rise to a $T^{1/4}$ law at low temperatures. At higher temperatures (energies) the nature of the system changes as the extended single-particle states become important. The conductivity should appear activated at higher temperatures. In principle the analog of Miller-Abrahams hopping may occur before activation in which case a change of shape in an $\ln\sigma$ vs T^{-1} plot should appear. The conclusion is based on perturbation theory and the convergence may alter the activation energy from its noninteracting value. We refer to activation energy rather than mobility edge, since the excited states are all mobile. The activation takes place due to a change in the nature of transport at some energy.

It should be clear, then, that insofar as a single-particle approach is valid, that interactions should not lead to deviations from a $T^{1/4}$ law. It is difficult to estimate the size of the pre-exponential, and it is not likely that an estimate based on Sec. V is very good. It seems likely that, just as in metal, the phonons are probably more effective than other electrons. A recent work[27] by Butcher and Swierkowski suggested otherwise.

Adkins[8] has suggested that a single-particle approach is not valid. In his model the electrons form a disordered version of a Wigner crystal, a Wigner glass. The activated transport is explained in terms of slipping and the anomalous Hall effect would not be so anomalous since, presumably, all of the electrons may participate. It is hard to see, however, how a $T^{1/4}$ law might come about in this model. It is our opinion that it is not necessary to be so extreme. We have seen how a quasiparticle picture will preserve the $T^{1/4}$ law with activated behavior at higher temperatures. If the interaction is long-range (or is effectively long-range due to broadening effects) we have seen how localized electrons may exchange energy with one another. Therefore, it may be possible that the electrons trapped below the mobility edge may not be frozen out of the picture as conventionally assumed, and a proper understanding of the Hall effect will necessarily

include interactions.

Before continuing, we should emphasize that these results, especially the $T^{1/4}$ law, depend on a smooth quasiparticle density of states. As mentioned in Sec. V this has been the subject of controversy. In terms of our formulation of the problem the well-behaved density depends on a regular $\mathrm{Re}S_\alpha(\omega)$. This quantity renormalizes the single-particle energies and thus the density. In order to obtain the density of states necessary to produce the $T^{1/2}$ behavior[20,21] the relevant quasiparticle energies ϵ must be related to ϵ_0 the bare particle energy by $\epsilon \sim \epsilon_0^{1/d}(d>1)$. In this case we easily obtain a $T^{1/2}$ law with our model. This is not surprising in view of our remarks in the Introduction. In fact, if we assume power law localization instead of exponential localization, we obtain results similar to those of Last and Thouless.[28] (It is necessary for the wave functions to decay more quickly than $r^{-d/2}$ in order to keep things finite.)

We have discussed the role of spin in terms of its effects on transport and magnetic properties. We have argued that the intrasite repulsion will not change the form of the $T^{1/4}$ law providing the system does not become metallic. This latter question may, perhaps, best be answered in terms of the m basis. Here, we hope, it is seen that the persistence of the Anderson transition in the presence of interactions is at least plausible.

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