Chapter 4

Neutral Fermions and Bosons

Our major interest in this work lies with the pure electromagnetic system: electrons, positrons, and the electromagnetic field. But the phenomena with which these are associated contain enough other interesing notions that a complete picture cannot be obtained without examining those notions. This chapter, therefore, is devoted to a number of peripheral subjects which, nevertheless, are essential to a full treatment of the Dirac theory.

A. Neutral Fermions

As mentioned earlier, a charge-neutral spin-1/2 particle can still interact with the electromagnetic field by means of the Pauli non-minimal coupling. In this case Eq.(3–11) becomes

$$\left(i\hbar\gamma^\mu\partial_\mu + \frac{1}{2}\frac{\mu}{c}\sigma^{\alpha\beta}F_{\alpha\beta}\right)\psi(\boldsymbol{r}) = mc\psi(\boldsymbol{r})\,, \tag{4–1}$$

where $\mu = a\mu_0$ is an anomalous magnetic moment. Although the neutron appears not to be elementary, but composite, it nevertheless is stable enough to warrant detailed study. Equation (4–1) therefore is taken to describe the motion of a free neutron in the presence of an electromagnetic field—at least in weak fields, so that the particle structure is unaffected. In this respect, we also note that the neutron possesses an electric polarizability (*e.g.*, Schmiedmayer, *et al*, 1988). Furthermore, should the neutrino turn out to have a small mass, as we shall discuss presently, then the presence of a small anomalous magnetic moment as well would lead to the conclusion that Eq.(4–1) is actually the equation of motion for the neutrino, which *is* an elementary particle.

In the following chapter we shall consider numerous applications of the Dirac theory, but it is useful to examine here one example of the use of the equation of motion (4–1). For the case of an external uniform magnetic field $\boldsymbol{B}$ the anomalous-moment coupling can be obtained from Eq.(3–9), so that Eq.(4–1) becomes

$$i\hbar\partial_t\psi = (c\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 + \mu\beta\boldsymbol{\Sigma}\cdot\boldsymbol{B})\psi\,. \tag{4–2}$$

Because there is no direct coupling between kinetic and magnetic energies in this problem ($\mu B \ll mc^2$), we can presume plane-wave solutions:

$$\psi_p(x) = \begin{pmatrix} w_1(\boldsymbol{p}) \\ w_2(\boldsymbol{p}) \\ w_3(\boldsymbol{p}) \\ w_4(\boldsymbol{p}) \end{pmatrix} e^{-ip\cdot x/\hbar}\,. \tag{4–3}$$

W. T. Grandy Jr., *Relativistic Quantum Mechanics of Leptons and Fields*

With the magnetic field in the positive z-direction, substitution of Eq.(4–3) into Eq.(4–2) yields a set of four linear algebraic momentum-space equations in four unknowns:

$$\begin{aligned}
&(mc^2 - E + \mu B)w_1 + cp_3w_3 + c(p_1 - ip_2)w_4 = 0\,,\\
&(mc^2 - E - \mu B)w_2 + c(p_1 + ip_2)w_3 - cp_3w_4 = 0\,,\\
&cp_3w_1 + c(p_1 - ip_2)w_2 - (mc^2 + E + \mu B)w_3 = 0\,,\\
&c(p_1 + ip_2)w_1 - cp_3w_2 - (mc^2 + E - \mu B)w_4 = 0\,.
\end{aligned} \tag{4–4}$$

These equations possess nontrivial solutions only if the determinant of coefficients vanishes. After a good deal of algebra this condition leads to the expression

$$(c^2\boldsymbol{p}^2 + m^2c^4 + \mu^2B^2 - E^2)^2 = 2\mu^2B^2(2m^2c^4 + 2c^2p_\perp^2)\,, \tag{4–5a}$$

where

$$p_\perp \equiv p\sin\theta \tag{4–5b}$$

is the magnitude of the projection of the momentum into the plane transverse to the field. The square-root yields factors ±1, which correspond to the two possible spin projections along the z-axis. Hence, the free-particle energies are given by

$$E^2(p,s) = c^2\boldsymbol{p}^2 + m^2c^4 + \mu^2B^2 + 2s\mu B(c^2p_\perp^2 + m^2c^4)^{1/2}\,, \tag{4–6}$$

and $s = \pm1$. Note that in the case of the neutron, for example, the anomalous magnetic moment is opposite to the spin, so that in our convention spin-up corresponds to $s = +1$. Finally, we see that E^2 is positive definite no matter what values are assigned to μ and B.

One might be tempted to examine the limit $m \to 0$ in Eq.(4–6) in the accepted case that the neutrino actually has a vanishing mass. This is not necessarily a straightforward procedure, however, owing to the rather peculiar properties of massless particles. Because the neutrino seems to be an absolutely elementary particle, therefore, it is in order to digress at this time to study some of these peculiarities.

Massless Particles

Recall that the Poincaré group contains the momentum operator P_μ as a generator of translations. With Wigner (1939) we introduce the notion of the *little group*. This is defined as a subgroup of Lorentz transformations leaving the momentum of a particle invariant:

$$\overline{\Lambda}_\mu^{\ \nu}\, p_\nu = p_\mu\,, \tag{4–7}$$

where p_ν is an eigenvalue of P_ν. For an infinitesimal transformation, $\overline{\Lambda}_{\mu\nu} = g_{\mu\nu} + \overline{\omega}_{\mu\nu}$, we obtain the condition

$$\overline{\omega}_{\mu\nu}\, p^\nu = 0\,, \tag{4–8}$$

which can be satisfied by an antisymmetric $\overline{\omega}_{\mu\nu}$. In the case of massive particles one finds that the little group is isomorphic to the orthogonal group O(3). Hence, the matrices of the rotation group are sufficient to describe the irreducible representations of the Poincaré group. We also recall the definition of helicity, Eq.(2–118), but note that massive particles do not necessarily have to be described by helicity states. Rather, they can be described by any linear combination of helicity states, and there are as many of these as there are spin states.

For massless particles, however, the situation is quite different. The little-group equation in this case is

$$\overline{\Lambda}_\mu^\nu k_\nu = k_\mu \,, \quad k_\mu \equiv (\kappa; 0, 0, \pm\kappa) \,, \tag{4–9}$$

say. The antisymmetric infinitesimal quantity now satisfies $\overline{\omega}_{\mu\nu} k^\nu = 0$, but possesses only three independent components. Wigner observed that in this case the little group is isomorphic to the Euclidean group in two dimensions, E(2), consisting of a rotation about the z-direction (here) and two 'translations'. The latter will be discussed later, and the rotation, he noted, is associated with the particle helicity. As usual, the helicity eigenvalues are given by

$$\lambda = \boldsymbol{J} \cdot \hat{\boldsymbol{k}} = \pm |J_3| \,, \tag{4–10}$$

so that massless particles with spin possess only *two* independent helicity states, no matter what the value of $(2S+1)$. We shall find these observations of particular value in studying the electromagnetic field.

Let us return to a study of the free-particle helicity states for spin-$\frac{1}{2}$ particles, Eq.(2–120), and consider the $(\frac{1}{2},0)$ spinor representation for a free massive particle of momentum $\boldsymbol{p}$ and helicity $\lambda = \pm 1$. In terms of rest-frame eigenspinors $\phi_\lambda^{(0)}(\boldsymbol{p})$, the covariantly normalized wavefunction is

$$\phi_\lambda(\boldsymbol{p}) = \frac{E + mc^2 + c\boldsymbol{\sigma} \cdot \boldsymbol{p}}{(E + mc^2)^{-1/2}} \, \phi_\lambda^{(0)}(\boldsymbol{p}) \,. \tag{4–11}$$

Note that there is a minus sign difference from the boosts of Eq.(2–44), because the latter actually refer to the Weyl representation. In the massless limit

$$\phi_\lambda(\boldsymbol{p}) \xrightarrow[m \to 0]{} E^{1/2}(1+\lambda)\phi_\lambda^{(0)} \,, \tag{4–12}$$

so that only the $\lambda = +1$ right-handed state survives for the $(\frac{1}{2},0)$ representation. Similarly, the replacements $\boldsymbol{p} \to -\boldsymbol{p}$, $\lambda \to -\lambda$, produce the $(0,\frac{1}{2})$ representation, for which only the $\lambda = -1$ left-handed state survives in the massless limit.

As a consequence of this analysis one concludes that massless spin-$\frac{1}{2}$ fermions can exist in only one definite helicity state, for each representation corresponds to a different type particle. That is, the particle must *always* be in one particular helicity state, either right-handed or left-handed, for there is no mass to couple the two. These observations are of fundamental importance to a study of the neutrino.

B. Theory of the Neutrino

Historically, the neutrino was introduced by Pauli in order to preserve the notion of energy conservation in β-decay. According to Carlson and Oppenheimer (1932), Pauli presented both the notion of a 'magnetic neutron' and the minimal-coupling equation (3–11) to describe it at a seminar in Ann Arbor during the summer of 1931. Indeed, they apparently were the first to apply this equation, and its subject was the neutrino (a name later coined by Fermi).

Originally the particle was to have spin $\frac{1}{2}$, zero charge, and zero mass, though Pauli suggested it might possess a magnetic moment and the effect of this on electron-neutrino scattering was considered by Bethe (1935). The aim, of course, was to find a fundamental interaction mechanism for the particle. Present experimental evidence suggests an extremely small moment, if any, the most recent results coming from analyses of the supernova 1987A: $(\mu/\mu_0) < 5 \times 10^{-13}$ (Lattimer and Cooperstein, 1988), and $(\mu/\mu_0) < (0.2 - 0.8) \times 10^{-11}$ (Barbieri and Mohapatra, 1988). But even moments *this* large are unexpected, and various models for an anomalous magnetic moment have been proposed (Fukugita and Yanagide, 1987; Babu and Mohapatra, 1989).

Both theoretically and experimentally the evidence is overwhelming that the neutrino charge is identically zero. Otherwise, because observable electric charge appears always in quantized units of the electronic charge e, there would exist strong experimental effects. Theoretical arguments that $m_\nu = 0$ are less compelling, however, although this is generally taken to be the case. But whether or not the neutrino mass is precisely zero has imporant consequences, particularly for astrophysics and cosmology. Although astronomy itself cannot provide direct evidence for a nonzero neutrino mass, the array of cosmological arguments for a finite mass are impressive (*e.g.*, Dolgov and Zeldovich, 1981). Current laboratory bounds on the neutrino mass are in the range 10-30 eV/c^2, and the entire subject has been reviewed recently (Boehm and Vogel, 1988). The relevance of finite mass to the solar-neutrino problem is also discussed in some depth by Weneser and Friedlander (1987; Friedlander and Weneser, 1987), and a general review of neutrino properties can be found in Klepdor (1988).

If the neutrino does not possess a magnetic moment, then it has no electromagnetic interaction at all. This motivates the hypothesis of a *weak* interaction and Fermi's point theory, currently subsumed into the so-called electroweak unification. Of course, if it eventually turns out that $m_\nu \neq 0$, then the notions of a magnetic moment and associated structure will become a bit more reasonable.

The Weyl Theory

Presume for the time being that $m_\nu \equiv 0$, as is conventional. Because the neutrino is a spin-$\frac{1}{2}$ fermion, it is clear that the Dirac equation immediately provides the equation of motion:

$$i\hbar\gamma^\mu\partial_\mu\psi(x) = 0\,. \tag{4–13}$$

The Dirac theory is relativistically covariant with or without mass. In order to understand this equation it is first useful to rewrite the theory for massive particles in an explicit 2-component form. With

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}, \tag{4–14}$$

the Dirac equation is equivalent to two coupled 2-component equations in the Pauli-Dirac representation:

$$\begin{aligned} (i\hbar\partial_t - mc^2)\phi &= c\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi\,, \\ (i\hbar\partial_t + mc^2)\chi &= c\boldsymbol{\sigma}\cdot\boldsymbol{p}\phi\,. \end{aligned} \tag{4–15}$$

Now write

$$\phi \equiv \varphi_R + \varphi_L\,, \qquad \chi \equiv \varphi_R - \varphi_L\,, \tag{4–16}$$

where the notation will be clarified presently. Equations (4–15) become

$$(i\hbar\partial_t - c\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_R = mc^2\varphi_L\,, \tag{4–17a}$$

$$(i\hbar\partial_t + c\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi_L = mc^2\varphi_R\,, \tag{4–17b}$$

which are familiar from Eqs.(2–63). One thus verifies that this *pair* of equations is invariant under the parity operation.

In the limit $m \to 0$ Eqs.(4–17) decompose into a set of *uncoupled* equations:

$$(i\hbar\partial_t - c\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_R = 0\,, \tag{4–18a}$$

$$(i\hbar\partial_t + c\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_L = 0\,, \tag{4–18b}$$

and the discussion concerning masless spin-$\frac{1}{2}$ particles following Eq.(4–9) is now relevant. That is, these equations correspond to separate representations of the Lorentz group, each of which describes a different type particle that can exist in only one definite helicity state. Hence, φ_R corresponds to a state of positive helicity and transforms according to the $(\frac{1}{2}, 0)$ irreducible representation of the Lorentz group. Similarly, φ_L corresponds to negative helicity and to the irreducible representation $(0, \frac{1}{2})$.

Equations (4–18) are more easily understood by changing to the Weyl representation of the γ-matrices introduced in Chapter 2. Although $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$ remains the same, now

$$\gamma^0 = \begin{pmatrix} 0 & -\sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \quad \boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}. \tag{4–19}$$

Effectively, γ^0 and γ^5 are interchanged in the two representations. In the Weyl representation Eqs.(4–18) are equivalent to having written the Dirac bispinor as

$$\psi = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}, \tag{4–20}$$

in analogy with Eqs.(4–14) and (4–15).

Under the parity transformation $\gamma_0 P_0$,

$$\begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} \longrightarrow \begin{pmatrix} -P_0\varphi_L \\ -P_0\varphi_R \end{pmatrix} . \tag{4–21}$$

Thus, under P Eqs.(4–18) transform into one another and neither by itself conserves parity. Even though we are not here considering charged particles, it is still possible to carry out the charge-conjugation transformation, and under C we find that

$$\begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix} \longrightarrow \begin{pmatrix} -i\sigma_2{\varphi_L}^* \\ \sigma_2{\varphi_R}^* \end{pmatrix} \equiv \begin{pmatrix} {\varphi'_R}^C \\ {\varphi_L}^C \end{pmatrix} , \tag{4–22}$$

as verified in the problems. Once again Eqs.(4–18) transform into one another but with the charge-conjugate spinors inserted, and so they are not individually covariant under C. They are, however, covariant under the combined operation CP, from whence came the notion that CP should be a strict symmetry of nature. One concludes again that it is the particle mass coupling the original equations that requires a 4-component description and conservation of parity, and that permits a mixing of helicity states to form arbitrary polarizations.

Now recall that γ^μ and γ^5 anticommute, so that if ψ is a solution of Eq.(4–13), then so is $\gamma^5\psi$. This is reminiscent of the discussion associated with Eqs.(3–49) and the notion of *chiral symmetry*, or 'handedness'. Consequently, one can impose on the solutions the additional constraints $\gamma^5\psi = \pm\psi$ so as to project from ψ the 2-component solutions. In the Weyl representation the chiral projection operators are

$$P_+ \equiv \frac{1}{2}(1+\gamma^5) = \begin{pmatrix} \sigma_0 & 0 \\ 0 & 0 \end{pmatrix} , \tag{4–23a}$$

$$P_- \equiv \frac{1}{2}(1-\gamma^5) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma_0 \end{pmatrix} . \tag{4–23b}$$

Then, if $\phi = P_-\psi$, we automatically have $\gamma^5\phi = -\phi$, and a free spin-$\frac{1}{2}$ fermion of zero mass is completely described by a 2-component spinor. That is, if we employ Eq.(4–20) and define

$$u \equiv P_+\psi = \begin{pmatrix} \varphi_R \\ 0 \end{pmatrix} , \tag{4–24a}$$

$$v \equiv P_-\psi = \begin{pmatrix} 0 \\ \varphi_L \end{pmatrix} , \tag{4–24b}$$

then the separate 2-component solutions are eigenfunctions of γ^5 with eigenvalues ± 1:

$$\gamma^5 u = u , \qquad \gamma^5 v = -v . \tag{4–25}$$

φ_L :

φ_R :

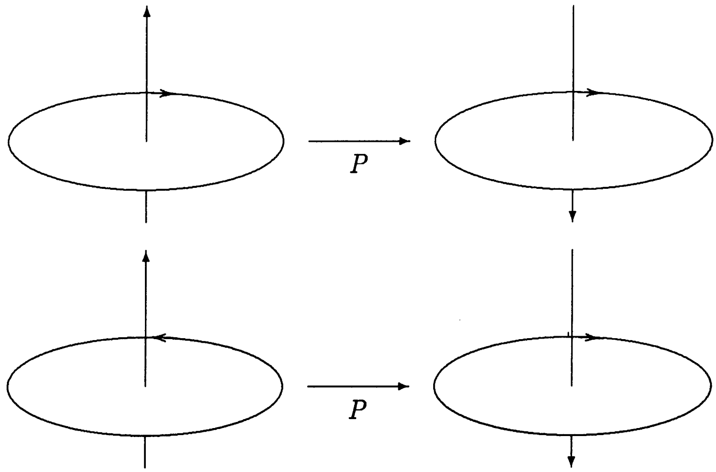

Fig. 4–1. (a) Neutrino and antineutrino states, and (b) their behavior under spatial inversion.

This is the Weyl theory (Weyl, 1929), which has been used to describe the neutrino since 1957—presumably because the description of a distinct particle by a single 2-component spinor makes manifest the non-conservation of parity (Lee and Yang, 1957; Landau, 1957; Salam, 1957).

Further insight emerges from an investigation of positive-energy plane-wave states,

$$\varphi(x) = \varphi(p)e^{-\frac{i}{\hbar}p\cdot x} . \tag{4–26}$$

Substitution into Eq.(4–18a) yields

$$p_0\varphi_R = (\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_R , \tag{4–27}$$

and multiplication by $(\boldsymbol{\sigma}\cdot\boldsymbol{p})$ provides the condition

$$(p_0^2 - \boldsymbol{p}^2)\varphi_R(p) = 0 . \tag{4–28}$$

The implication is that nonvanishing solutions exist only if $p_0 = \pm|\boldsymbol{p}|$, so that massless neutrinos travel at the speed of light. Thus, one concludes that

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\varphi_R = \varphi_R \tag{4–29a}$$

describes a positive-energy right-handed particle, whereas the plane-wave solution to Eq.(4–18b),

$$(\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}})\varphi_L = -\varphi_L , \tag{4–29b}$$

decribes a positive-energy left-handed particle. The pseudo-scalar character of $(\boldsymbol{\sigma}\cdot\boldsymbol{p})$ under P is illustrated in Figure 4–1. Note that $\varphi_R(\boldsymbol{p})$ is also proportional to a negative-energy solution $\varphi_L(-\boldsymbol{p})$.

If we examine the 4-component plane-wave solution for positive energy we are led to the expression $\gamma^0 p_0 \psi = (\boldsymbol{\gamma} \cdot \boldsymbol{p})\psi$. Because $\gamma^5\gamma^0$ is the same in either the Pauli-Dirac or Weyl representation, multiplication by $\gamma^5\gamma^0$ yields the relation

$$\gamma^5 p_0 \psi = \boldsymbol{\Sigma} \cdot \boldsymbol{p}\psi\,, \tag{4–30}$$

in either representation. But we know that $p_0 = |\boldsymbol{p}|$, so that

$$\gamma^5 \psi = \boldsymbol{\Sigma} \cdot \hat{\boldsymbol{p}}\psi\,, \tag{4–31}$$

and chirality and helicity are one and the same thing here.

The complete set of free-particle equations is, of course, invariant under the full Lorentz group, and cannot by itself imply anything about non-conservation of parity. Such asymmetries can only arise from an interaction mechanism, as with β-decay, or as a consequence of initial conditions, as noted earlier. In this latter event, one can think of the neutrino as polarizing the interaction. By choosing a single polarization to represent a particle we define the Weyl theory. Experiment—and that alone—dictates that Eq.(4–29b) describes the neutrino, which is left-handed and has its spin antiparallel to its momentum. In this sense, Eq.(4–29a) describes an antineutrino. Of course, this chirality (or 'handedness') has long been known elsewhere in nature—for example, in the optical activity of sugars and amino acids. More recently, the discovery of parity-violating weak neutral currents has led to direct evidence for chirality in individual atoms (*e.g.*, Hegstrom, *et al*, 1988).

As an interesting aside, consider the current operator $j_\mu = \overline{\psi}\gamma_\mu\psi$. It is conventional in this context to omit the factors of 1/2 from the chiral projection operators, so that in momentum space the neutrino current is

$$\begin{aligned}\langle \boldsymbol{p}'|j_\mu^{(\nu)}|\boldsymbol{p}\rangle &= \overline{\varphi_L}\gamma_\mu\varphi_L \\ &= \overline{v}(1+\gamma_5)\gamma_\mu(1-\gamma_5)v\,,\end{aligned} \tag{4–32}$$

in the notation of Eqs.(4–24). With the identities

$$P_\pm^2 = P_\pm\,, \qquad P_+P_- = 0\,, \tag{4–33a}$$

$$\gamma_\mu(1-\gamma_5) = (1+\gamma_5)\gamma_\mu\,, \tag{4–33b}$$

we can rewrite this matrix element as

$$\langle \boldsymbol{p}'|j_\mu^{(\nu)}|\boldsymbol{p}\rangle = \overline{v}(\boldsymbol{p}')\gamma_\mu(1-\gamma_5)v(\boldsymbol{p})\,. \tag{4–34a}$$

Similarly, for the antineutrino we have

$$\langle \boldsymbol{p}'|j_\mu^{(\overline{\nu})}|\boldsymbol{p}\rangle = -\overline{u}(\boldsymbol{p})\gamma_\mu(1+\gamma_5)u(\boldsymbol{p}')\,, \tag{4–34b}$$

in which we note a reversal of momenta expected from C-invariance. One again encounters a minus sign illustrating Feynman's view of antiparticles as particles travelling backward in time.

It is possible to consider the neutrino in the presence of an electromagnetic field, but clearly the coupling cannot take place by means of a charge and a minimal replacement. Indeed, from Eq.(3–106) we see that the electromagnetic current can at best have the form

$$\langle \boldsymbol{p}'|J_\mu^{(\nu)}|\boldsymbol{p}\rangle \sim F_1(q^2)\overline{v}(\boldsymbol{p}')\gamma_\mu(1-\gamma_5)v(\boldsymbol{p})\,, \tag{4–35}$$

where necessarily $F_1(0) = 0$. That is, $F_2 = 0$, because the 2-component neutrino can have no magnetic moment, not even a form factor. This last observation follows from noting that the term normally containing $F_2(q^2)$ also contains a term $\sigma^{\alpha\beta}q_\beta v$, with $q \equiv p' - p$, and this vanishes identically owing to the Dirac equation (4–13). Physically, the presence of a magnetic moment would require the spin to be flipped in certain interactions, but if the neutrino has only one spin direction it cannot be flipped.

If $m_\nu \neq 0$, then the neutrino is described by a 4-component bispinor satisfying Eq.(4–1), including a possible anomalous magnetic moment. But even if $m_\nu = 0$ this equation cannot split into a 2-component Weyl theory if $\mu \neq 0$, because the anomalous moment couples the components in the presence of a field. Moreover, the conserved currents $\overline{\psi}\gamma^\mu\psi$ and $\partial_\nu(\overline{\psi}\sigma^{\mu\nu}\psi)$ are no longer proportional by means of Gordon's decomposition. Thus, if the neutrino possesses an anomalous magnetic moment it must be described by a 4-component wavefunction—which *may* reduce to 2-component form asymptotically distant from the interaction.

The Four-Component Neutrino

A 4-component theory of the neutrino with $m_\nu \neq 0$ was constructed by Majorana (1937), who appended to the Dirac equation the constraint $\psi_C = \psi$. That is, the particle is self-charge-conjugate, so that there is no distinction between particle and antiparticle. This condition is most readily realized by reference to Eq.(4–22), in the Weyl representation, from which we obtain

$$\psi = \begin{pmatrix} -i\sigma_2\varphi^* \\ \varphi \end{pmatrix} = \begin{pmatrix} \varphi_C \\ \varphi \end{pmatrix}. \tag{4–36}$$

The coupled 2-component equations in this representation are then

$$\begin{aligned} i\hbar\partial_t\varphi + c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi &= -mc^2\varphi_C\,, \\ i\hbar\partial_t\varphi_C - c(\boldsymbol{\sigma}\cdot\boldsymbol{p})\varphi_C &= -mc^2\varphi\,. \end{aligned} \tag{4–37}$$

Once again, we know that the full set of equations is invariant under the parity transformation, so that any asymmetries in weak interactions must be dynamical if the particles involved are Majorana neutrinos. It is clear that in the limit $m \to 0$ this theory goes over into the 2-component Weyl theory. In fact, there is no way to distinguish between Weyl neutrinos and the $(m=0)$-limit of Majorana neutrinos, except for the condition $\psi_C = \psi$. Detailed comparisons between the two theories have been provided by Serpe (1952), McLennan (1957), and Case (1957).

There is actually a process that can test these possibilities. If β-decay to the next element in the Periodic Table is energetically forbidden, or otherwise inhibited, a jump to the next-nearest element might be allowed by double-beta decay. That is, two neutrons within a nucleus might simultaneously decay via $n \rightarrow p + e^- + \overline{\nu}$, and the two Dirac antineutrinos detected. It is very difficult to detect this process, for the decay rate is exceedingly weak. Nevertheless, the decay mode was finally observed directly in 1987, from a ^{82}Se source (Elliott, *et al*, 1987). But if the neutrinos are Majorana with a small mass, then $\overline{\nu} = \nu$ and we can also consider the process $n + \nu \rightarrow p + e^-$. A virtual neutrino could be emitted by the first neutron and absorbed by the second, resulting in *neutrinoless* double-beta decay (*e.g.*, $^{124}\text{Sn} \rightarrow {}^{124}\text{Te}+2e^-$). Observation of this mode could resurrect the Majorana theory—as well as suggest a violation of lepton conservation. The neutrinoless mode has yet to be observed, but with the successful observation of the two-neutrino decay efforts have increased. A recent review of double-beta decay is provided by Rosen (1988).

Alternatively, let us take $m_\nu = 0$ and explore the consequences of an anomalous magnetic moment for the neutrino, which must also be described by a 4-component bispinor. (Owing to charge-conjugation invariance, this cannot be a Majorana neutrino.) To be specific, consider the general nonminimal equation (3–11), which we take to describe a charged Dirac paricle with anomalous moment in the presence of a fixed Coulomb field. As is shown in detail in the following chapter, spherical symmetry persists and the coupled radial equations generalizing those of Eq. (3–42) are

$$\frac{df}{dr} - \frac{\kappa - 1}{r} + \left(W - \lambda_c^{-1} - \frac{Z\alpha}{r}\right) g = \frac{1}{r^2} a' \frac{Ze^2}{2mc^2} f\, F(r)\,, \tag{4–38a}$$

$$\frac{dg}{dr} + \frac{\kappa + 1}{r} g - \left(W + \lambda_c^{-1} - \frac{Z\alpha}{r}\right) f = -\frac{1}{r^2} a' \frac{Ze^2}{2mc^2} g\, F(r)\,, \tag{4–38b}$$

where $F(r)$ is a magnetic form factor, and a' is the anomaly. With Barut (1980), we define the *neutrino limit*:

$$e \rightarrow 0\,, \qquad m \rightarrow 0\,, \qquad a' \tfrac{\hbar e}{2mc} \rightarrow \mu = a\mu_0\,, \tag{4–39}$$

where a may be different than a', and the e of the Coulomb source is unaffected. The coefficients on the right-hand sides of Eqs.(4–38) are then

$$V_m(r) \equiv \frac{e\mu}{\hbar c} \frac{F(r)}{r^2}\,, \tag{4–40}$$

and the terms in λ_c^{-1} and $Z\alpha$ vanish.

These equations for the neutrino with anomalous moment in a Coulomb field are very difficult to analyze, of course. The general procedure is to convert them to uncoupled second-order equations, as in Chapter 5, and for $\epsilon \equiv \text{sgn}(e\mu) = -1$ the effective potentials will exhibit deep magnetic wells. A qualitative picture can be developed, however, by restricting our interest to zero-energy solutions. The first-order equations (4–38) then become

$$\frac{df}{dr} = \left(\frac{\kappa - 1}{r} + V_m\right) f\,, \tag{4–41a}$$

$$\frac{dg}{dr} = -\left(\frac{\kappa + 1}{r} + V_m\right) g\,, \tag{4–41b}$$

and only in this case do the equations decouple. Under this restriction the system mass of a bound state or resonance will be just that of the central particle.

As is readily verified, the general solutions to Eqs.(4–41) are

$$f(r) = C_1 r^{\kappa-1} \exp\left\{\int_0^r V_m(r)\,dr\right\}\,, \tag{4–42a}$$

$$g(r) = C_2 r^{-(\kappa+1)} \exp\left\{-\int_0^r V_m(r)\,dr\right\}\,, \tag{4–42b}$$

the C_i being constants. Note that the coefficient $(e\mu/\hbar c)$ in Eq.(4–40) has the dimension of length—in fact, it is $a/2$ times the classical electron radius. We shall label this coefficient r_0, and rewrite Eq.(4–40) as

$$V_m(r) = \frac{\epsilon r_0}{r^2} F(r)\,. \tag{4–43}$$

Although a complete integration requires knowledge of the close-in behavior of the form factor, at this stage we are only interested in the behavior of the wave-functions for $r > r_0$. Consequently, it suffices to presume simply that F vanishes strongly at the origin and is unity for $r > r_0$. Then, if F is otherwise almost constant,

$$f(r) = C_1 r^{\kappa-1}\, e^{-\epsilon r_0/r}\,, \tag{4–44a}$$

$$g(r) = C_2 r^{-(\kappa+1)}\, e^{\epsilon r_0/r}\,. \tag{4–44b}$$

As an example, one could have

$$F(r) \propto 1 - e^{-2r/r_0}\left[1 + (2r/r_0)^n\right]\,, \qquad n \geq 1\,. \tag{4–44c}$$

The qualitative conclusions to be drawn from these results are rather interesting. For example, if $\kappa = 1$, $\epsilon = -1$, then $f(r)$ 'leaks out' and approaches a constant as $r \to \infty$, whereas $g(r)$ is localized around r_0 and is normalizable. For $\kappa = -1$, $\epsilon = +1$, we obtain the opposite behavior. These solutions are of some interest because neutrinos in asymptotic states seem to be described naturally by 2-component spinors, so that these asymptotic states are *not* eigenstates of parity. A full understanding of this phenomenon, of course, can only come from a complete study of the actual two-body problem.

As an aside, we note that even if $m_\nu \neq 0$ the equations of motion can still be integrated for zero-energy solutions. In this case $E = mc^2$ and we still obtain the same equation for f. That for g, however, becomes

$$\frac{dg}{dr} = \left(-\frac{\kappa + 1}{r} - V_m\right) g + 2\lambda_c^{-1} f\,. \tag{4–45}$$

Nevertheless, as is demonstrated in Problem 4.2, the qualitative picture is essentially unchanged.

Lepton Families

Let us digress for a moment for a few remarks concerning leptons in general. The only members in this category of elementary particles that we have discussed in any detail are the electron (and to some extent the positron), and the neutrino (and $\overline{\nu}$). They have been treated absolutely as point particles with no observable structure. (Although the name 'lepton' has been used for 1/100 of a Greek drachma, it is the connotation of 'light one' that applies here!) It is useful to think of e^- and ν (along with e^+ and $\overline{\nu}$) as a single family, for it turns out that there are other families of leptons.

The *muon* was discovered by Anderson and Neddemeyer (1937), and Street and Stevenson (1937) in cosmic-ray searches for Yukawa's pi meson (pion), thought to be the quantum of the nuclear force. Indeed, it was first thought to be the pion, but this notion soon proved wrong and the particle was then called a mu meson. Upon evidence that it was by no means a meson, the simple name muon was adopted. Although the muon mass is some 207 times the mass of the electron, and it is unstable with a lifetime of 2.2×10^{-6}sec, in all other respects it is much like an electron. The charge of μ^- is the same as that of the electron (μ^+ is the antiparticle), it has spin $\frac{1}{2}$ with a small anomalous magnetic moment, there is no evidence for internal structure, and its radius appears to be less than 10^{-16}cm. The primary decay mode (98.6%) is to $e\nu\overline{\nu}$, and the electromagnetic decay cross section for $\mu \to e+\gamma$ is some 10^{-10} smaller than this. Thus, the muon cannot be merely an excited state of the electron in the atomic sense. But the exact role of the particle in the scheme of things remains an open question.

Neutrinos produced under ordinary β-decay, $n \to p+e^-+\overline{\nu}$, can be employed to study the inverse process $\overline{\nu}+p \to n+e^+$. Over twenty-five years ago it also became possible to produce copious amounts of neutrinos from the pion decay $\pi \to \mu+\nu$, but when these neutrinos were employed in the above processes one found $\nu+n \to p+\mu^-$ and $\overline{\nu}+p \to n+\mu^+$—with *no electrons*. In some way all the neutrinos here were associated only with muons, and since then it has been customary to distinguish between ν_e and ν_μ, whose only known difference is this 'electronness' and 'muonness' exhibited in processes like those above. One can now understand why the dominant decay must be $\mu^- \to e^-+\overline{\nu}_e+\nu_\mu$, say, preventing annihilation of the neutrinos to a γ. Reviews of the general two-neutrino experiment have been provided by Schwarz (1989), Steinberger (1989), and Lederman (1989) in their Nobel lectures.

In the mid-seventies a third charged lepton, the tau, was discovered, and an associated neutrino, ν_τ, inferred (Perl, *et al*, 1975). The primary decay mode is $\tau^- \to \mu^-+\overline{\nu}_\mu+\nu_\tau$, say, with lifetime $\sim 10^{-13}$sec. The mass of the tau is about 1 785 MeV, which now brings the aptness of the name 'lepton' into question! Nevertheless, except for mass and lifetime it exhibits all the leptonic features common to electrons and muons, including a corresponding 'tauness'. As far as is known, the three sets of lepton numbers are separately conserved in all processes.

An obvious question is whether these pairs are just the beginning of a sequence: (e,ν_e), (μ,ν_μ), (τ,ν_τ), $\cdots$. The question is one of more than mere curiosity, for the answer has potentially large astrophysical consequences (*e.g.*, Dolgov and Zeldovich, 1981). From those considerations it appears that such a sequence cannot be much longer than currently known—indeed, there is strong evidence that there are only three families (ALEPH Collaboration, 1990), though there is no fundamental explanation for why. Whatever the number, presumably all that has gone before, and all that comes later, applies to all lepton families. We shall, however, have more to say about muons in Chapter 8.

C. Boson Wave Equations

At one time it was thought that it may be possible to develop a *neutrino theory of light*, so that if one adopts a quantized particulate view of the electromagnetic field the photon would be a composite particle. The basic ideas were generated by de Broglie, Jordan, and Kronig, and the history is reviewed at length by Pryce (1938). Indeed, it was Pryce who showed in detail why the theory must fail, the essential cause being that light waves are polarized transversely, whereas neutrinos are polarized longitudinally. As we have seen, group-theoretical arguments show the impossibility of constructing the former from the latter.

Thus, we are led to to think of the photon as the most fundamental stable boson—if, indeed, one is inclined to adopt a particulate view of the electromagnetic field to begin with. Although we shall adduce arguments against the quantized field in subsequent chapters, it is nevertheless instructive to pursue the conventional path for the moment. There are, of course, other bosons that live long enough to provide interesting phenomena, such as in π-mesic atoms, so it is useful to discuss briefly some of the relevant equations governing their behavior.

Spin-0 Bosons

The manifestly covariant wave equation for spin-0 bosons is just the Klein-Gordon equation,

$$\left(\frac{1}{c^2}\partial_{tt}^2 - \nabla^2 + \lambda\!\!\!^{-}_c{}^{-2}\right)\psi(x) = 0\,, \tag{4–46}$$

where ψ is either a scalar or a pseudoscalar. We have noted earlier that the probability density is not positive-definite, negative-energy states arise, and *Zitterbewegung* is present. It is possible, however, to construct a consistent one-particle theory at low energies by interpreting ρ and $\boldsymbol{j}$ as charge density and electromagnetic current density, respectively. One notes that upon making the minimal coupling replacement Eq.(4–46) actually possesses two degrees of freedom, corresponding not only to positive and negative energies, but also to charges $\pm Q$. These can describe $\pi^{\pm}$, say, and if the wavefunction is taken to be purely real or imaginary the probability density vanishes, as would be expected for a charge-neutral particle. It might also be expected from these observations that a 2-component formulation would be rather effective, and this has been provided by Feshbach and Villars (1958).

Let ψ satisfy the Klein-Gordon equation (4–46) and define two new functions in terms of the time derivative, $\dot{\psi}$:

$$\varphi \equiv \frac{1}{2}\left(\psi + i\frac{\hbar}{mc^2}\dot{\psi}\right), \qquad \chi \equiv \frac{1}{2}\left(\psi - i\frac{\hbar}{mc^2}\dot{\psi}\right). \tag{4–47}$$

Noting that $\psi = (\varphi + \chi)$, $\dot{\psi} = (mc^2/i\hbar)(\varphi - \chi)$, and $\ddot{\psi} = c^2(\nabla^2 - \lambda_c^{-2})\psi$, we find the following equations of motion:

$$i\hbar\partial_t\varphi = -\tfrac{\hbar^2}{2m}\nabla^2(\varphi + \chi) + mc^2\varphi\,, \tag{4–48a}$$

$$i\hbar\partial_t\chi = \tfrac{\hbar^2}{2m}\nabla^2(\varphi + \chi) - mc^2\chi\,. \tag{4–48b}$$

The 2-dimensional state vector

$$\Phi \equiv \begin{pmatrix} \varphi \\ \chi \end{pmatrix} \tag{4–49}$$

then satisfies the Schrödinger-like equation

$$i\hbar\partial_t\Phi = \mathsf{H}\Phi\,, \tag{4–50a}$$

with

$$\mathsf{H} \equiv (\sigma_3 + i\sigma_2)\frac{\boldsymbol{p}^2}{2m} + mc^2\sigma_3\,. \tag{4–50b}$$

Although this 'Hamiltonian' is not Hermitian, the resulting theory can nevertheless be made consistent. Minimal coupling with charge Q leads also to a charge-conjugate wavefunction Ψ_C associated with charge $(-Q)$. An elegant feature of this formalism is that in the nonrelativistic limit $\chi \to 0$ and and one obtains the Schrödinger equation for φ corresponding to charge Q and positive energies.

Plane-wave solutions can be found in the form

$$\Psi = \Psi_0(\boldsymbol{p})e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x} - Et)} = \begin{pmatrix} \varphi_0(\boldsymbol{p}) \\ \chi_0(\boldsymbol{p}) \end{pmatrix} e^{\frac{i}{\hbar}(\boldsymbol{p}\cdot\boldsymbol{x} - Et)}\,, \tag{4–51}$$

yielding stationary-state equations

$$(E - mc^2)\varphi_0 = \frac{\boldsymbol{p}^2}{2m}(\varphi_0 + \chi_0)\,, \tag{4–52a}$$

$$(E + mc^2)\chi_0 = -\frac{\boldsymbol{p}^2}{2m}(\varphi_0 + \chi_0)\,. \tag{4–52b}$$

One verifies the solutions to be

$$\Psi_0^{(+)}(\boldsymbol{p}) = \begin{pmatrix} \varphi_0^{(+)} = \frac{E_p + mc^2}{2(mc^2E_p)^{1/2}} \\ \chi_0^{(+)} = \frac{mc^2 - E_p}{2(mc^2E_p)^{1/2}} \end{pmatrix}\,, \quad E = E_p\,, \tag{4–53a}$$

$$\Psi_0^{(-)}(\boldsymbol{p}) = \begin{pmatrix} \varphi_0^{(-)} = \frac{mc^2 - E_p}{2(mc^2E_p)^{1/2}} \\ \chi_0^{(-)} = \frac{E_p + mc^2}{2(mc^2E_p)^{1/2}} \end{pmatrix}\,, \quad E = -E_p\,, \tag{4–53b}$$

with energies

$$E = \pm E_p = \pm\left[(c\boldsymbol{p})^2 + (mc^2)^2\right]^{1/2}. \tag{4–54}$$

In both cases normalization is given by $\varphi_0^2 - \chi_0^2 = 1$. These are indeed charge-conjugate solutions, even in the absence of external fields, and expectation values are defined as

$$\langle A \rangle \equiv \int \Psi^\dagger \sigma_3 A \Psi \, d^3x \,. \tag{4–55}$$

The adjoint operator is defined so as to preserve charge states:

$$\overline{A} \equiv \sigma_3 (\widetilde{A})^* \sigma_3 \,. \tag{4–56}$$

When minimal coupling is included it is found useful to represent the state vector in the plane-wave basis defined by Eq.(4–54):

$$\Psi(\boldsymbol{p}, t) = u(\boldsymbol{p}, t)\Psi_0^{(+)}(\boldsymbol{p}) + v(\boldsymbol{p}, t)\Psi_0^{(-)}(\boldsymbol{p}) \,, \tag{4–57}$$

where u and v are scalar amplitudes. Feshbach and Villars have studied the effects of weak electric and magnetic fields on the spin-0 particle, and in such fields one finds that quite generally $v \ll u$. For example, in a Coulomb field

$$\frac{v}{u} \sim \frac{1}{8}(Z\alpha)^4 \,. \tag{4–58}$$

The 2-component formalism has proved quite effective in studying π-mesic atoms in weak fields. As a further example, consider a weak external uniform magnetic field $\boldsymbol{B}$, such that $\boldsymbol{A}(\boldsymbol{x}) = \frac{1}{2}(\boldsymbol{B} \times \boldsymbol{r})$. Then,

$$i\hbar \partial_t u = E_p u - \frac{e}{2mc}\left(\frac{mc^2}{E_p}\right) \boldsymbol{B} \cdot (i\hbar \nabla_p \times \boldsymbol{p}) u(\boldsymbol{p}) \,. \tag{4–59}$$

But in momentum space the orbital angular momentum is $\boldsymbol{L} = i\hbar\nabla_p \times \boldsymbol{p}$, so that we can define an effective magnetic moment

$$\boldsymbol{\mu} \equiv \frac{e}{2mc}\left(\frac{mc^2}{E_p}\right) \boldsymbol{L} \,, \tag{4–60}$$

and Eq.(4–59) acquires a familiar look. One sees that in π-mesic atoms relativistic effects tend to *reduce* the Zeeman splitting.

Spin-1 Bosons

In this case the wavefunction is a 3-component object $\phi_\mu = (0, \phi_i)$, such that it is either vector or pseudovector. From this one can also construct a second-rank antisymmetric tensor

$$\phi_{\mu\nu} \equiv \partial_\mu \phi_\nu - \partial_\nu \phi_\mu \,, \tag{4–61}$$

where $\phi_{0i} = -\phi_{i0} = c^{-1}\partial_t \phi_i$, and $\phi_{00} = \phi_{ij} = 0$. The corresponding dynamical equation for free particles is called the *Proca equation* (Proca, 1936a,b; Kemmer, 1939):

$$\partial_\nu \phi^{\mu\nu} = \lambda\!\!\!^{-}{}_c^{-2} \, \phi^\mu \,, \tag{4–62}$$

and Eqs.(4–61) and (4–62) imply the subsidiary condition

$$\partial_\mu \phi^\mu = 0 . \tag{4–63}$$

All of these expressions can then be combined to yield a Klein-Gordon equation for the vector amplitude:

$$(\Box + \lambda\!\!\!^{-}_c{}^{-2})\phi_\mu(x) = 0 , \tag{4–64}$$

which guarantees the correct energy-momentum relation for a free particle.

This Proca-Kemmer theory may possibly describe objects like the ρ and ω mesons. It has also been used to study the possibility of a nonzero photon mass by means of geophysical data (*e.g.*, Goldhaber and Nieto, 1971). That is, a stationary static charge at the origin will be described, in minimal coupling, by $\boldsymbol{A} = 0$ and

$$A_0 = Q\frac{e^{-\mu r}}{r} , \qquad \mu \equiv \lambda\!\!\!^{-}_\gamma{}^{-1} . \tag{4–65}$$

The validity of Coulomb's law is then related to the photon Compton wavelength, λ_γ.

The Photon

Although we are not particularly commited to quantization of the electromagnetic field at this point, as mentioned above, it is useful nevertheless to think occasionally of the photon as a particle—a spin-1 boson. This is by no means necessary here, but it does provide an opportunity to incorporate the electromagnetic field into the present formalism, and in a coherent way. We shall return to further discussion of this matter in Chapter 7.

It is almost certain that $m_\gamma = 0$. A photon with spin 1 is described by a vector potential A_μ and field tensor $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. Equation (4–62) provides the wave equation for free photons:

$$-\partial^\nu F_{\mu\nu} = \Box A_\mu - \partial_\mu(\partial_\nu A^\nu) = 0 , \tag{4–66}$$

so that A_μ serves as a wavefunction. Note that the subsidiary condition (4–63) is no longer automatic, but must be specified in accordance with the choice of helicity constraint.

Recall that for $m \neq 0$ there are now three helicity states for spin-1 particles, and these can be described by three mutually orthogonal polarization vectors. Right- and left-handed, transverse, circularly polarized states correspond to $\epsilon_\mu^{(\pm)}(\boldsymbol{p})$, and $\boldsymbol{\epsilon}^{(0)}(\boldsymbol{p})$ is a longitudinal polarization vector. But we have seen that the photon can have only two helicity states. Unlike the case for spin-$\frac{1}{2}$ particles, though, these states are not completely dictated by covariance arguments.

A free photon can be described by a plane wave satisfying Eq.(4–66):

$$A_\mu(x) = \epsilon_\mu(\boldsymbol{k})e^{-ik\cdot x} . \tag{4–67}$$

The subsidiary condition defining this representation is

$$k^\mu \epsilon_\mu^{(\pm)}(\boldsymbol{k}) = 0\,, \tag{4–68}$$

and can be used to ensure that there are only two independent components of ϵ_μ. This condition is not sufficient, but we can use the momentum vector of Eq.(4–9) to choose a *transverse gauge* by writing

$$\epsilon_0^{(\pm 1)}(\boldsymbol{k}) = 0\,, \qquad \boldsymbol{k}\cdot\boldsymbol{\epsilon}^{(\pm 1)}(\boldsymbol{k}) = 0\,. \tag{4–69}$$

The covariant choice (4–68) then corresponds to the Lorentz gauge:

$$\partial^\nu A_\nu = 0\,, \tag{4–70a}$$

so that the wave equation (4–66) takes the familiar form

$$\Box A_\mu = 0\,. \tag{4–70b}$$

The electromagnetic field tensor $F_{\mu\nu}$ is gauge invariant, of course, and the general dynamical equations are just

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\nu\,, \tag{4–71a}$$

or

$$\Box A^\mu(x) = \frac{4\pi}{c} j^\mu(x)\,. \tag{4–71b}$$

Because $F^{\mu\nu} = -F^{\nu\mu}$, it then follows that the current is conserved:

$$\partial_\mu j^\mu = 0\,. \tag{4–72}$$

Unfortunately, the two conditions of Eq.(4–69) cannot be chosen separately in a frame-independent manner. Alternatively, a Lorentz transformation perpendicular to the photon momentum will not maintain the timelike component A_0 equal to zero, for the transverse gauge condition is *not* manifestly covariant. It has been shown by Weinberg (1964a,b), however, that the Lorentz-transformed 4-potential can be made purely transverse if it is followed by a gauge transformation, and this gauge transformation is just that part of the little group associated with the 'translational' degrees of freedom. By definition the little group leaves k_μ invariant, so that the gauge transformation has no effect on the prior Lorentz transformation. Han and Kim (1981; Han, *et al*, 1985) have provided a detailed discussion of this procedure, which we outline briefly.

Consider a photon moving in the positive z-direction with frequency ω, such that its energy-momentum 4-vector can be written $k^\mu = \omega(1,0,0,1)/c$. In this case the little group can be represented by the matrix

$$D_\nu^\mu(\theta,u,v) = D(0,0,v)D(0,u,0)D(\theta,0,0)\,, \tag{4–73a}$$

where

$$D(0,0,v) = \begin{pmatrix} 1+v^2/2 & 0 & v & -v^2/2 \\ 0 & 1 & 0 & 0 \\ v & 0 & 1 & -v \\ v^2/2 & 0 & v & 1-v^2/2 \end{pmatrix}, \tag{4–73b}$$

$$D(0,u,0) = \begin{pmatrix} 1+u^2/2 & u & 0 & -u^2/2 \\ u & 1 & 0 & -u \\ 0 & 0 & 1 & 0 \\ u^2/2 & u & 0 & 1-u^2/2 \end{pmatrix}, \tag{4–73c}$$

$$D(\theta,0,0) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4–73d}$$

To be more specific, we take the polarization along the positive x-axis, so that the plane-wave vector potential is

$$A^\mu(x) = (0,1,0,0)e^{i\frac{\omega}{c}(z-ct)}. \tag{4–74}$$

By definition $D(0,u,0)$ is a Lorentz transformation. That it is also a gauge transformation follows from the observation that

$$\begin{aligned} G^\mu(x) &\equiv D(0,u,0)A^\mu(x) \\ &= \begin{pmatrix} u \\ 1 \\ 0 \\ u \end{pmatrix} e^{i\frac{\omega}{c}(z-ct)}, \end{aligned} \tag{4–75}$$

and the definitions

$$\begin{aligned} \boldsymbol{A} &\longrightarrow \boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda, \\ A_0 &\longrightarrow A_0' = A_0 - \tfrac{1}{c}\partial_t\Lambda, \end{aligned} \tag{4–76}$$

which define a gauge transformation of the second kind. That is, we identify

$$\Lambda = \tfrac{cu}{i\omega}e^{i\frac{\omega}{c}(z-ct)}, \tag{4–77}$$

and verify Eqs.(4–76).

If we now make a Lorentz transformation to a (primed) frame moving in the positive x-direction with velocity $\boldsymbol{v}$, we can use Eq.(2–3) to obtain the transformed quantities

$$k'^\mu = \frac{\omega}{c}\begin{pmatrix} \gamma \\ -\beta\gamma \\ 0 \\ 1 \end{pmatrix}, \qquad A'^\mu = \begin{pmatrix} -\beta\gamma \\ \gamma \\ 0 \\ 0 \end{pmatrix}, \tag{4–78a}$$

$$G'^{\mu} = \begin{pmatrix} \gamma(u-\beta) \\ \gamma(1-\beta u) \\ 0 \\ u \end{pmatrix} . \tag{4–78b}$$

Thus, in general $A'_0 \neq 0$. But if we apply a gauge transformation with $u = \beta$ *before* the Lorentz transformation, then the conditions $k^\mu A_\mu = 0$, $\boldsymbol{k} \cdot \boldsymbol{A} = 0$ are clearly maintained: $K'^\mu G'_\mu = 0$, $\boldsymbol{k}' \cdot \boldsymbol{G}' = 0$. Among other things, this exercise provides a physical interpretation for the parameters of the little group.

Problems

4.1 Calculate C in the Weyl representation and identify $i\sigma_2 K$ as the 2-dimensional version of the charge-conjugation operator. Demonstrate the C-invariance of the Weyl theory by showing, for example, that

$$(i\hbar\partial_t + c\boldsymbol{\sigma} \cdot \boldsymbol{p}(-i\sigma_2\varphi_L^*) = 0 .$$

4.2 With $m_\nu \neq 0$, show that Eq.(4–45) yields

$$g(r) = r^{-(\kappa+1)}\, e^{\epsilon r_0/r} \left[C_2 + 2\lambda_c^{-1} C_1 \int_0^r x^{2\kappa}\, e^{-2\epsilon r_0/x}\, dx \right] ,$$

and evaluate the integral for $\epsilon = +1$ and $r > r_0$, thereby obtaining behavior qualitatively similar to Eq.(4–44b).

REFERENCES

ALEPH Collaboration, D.Decamp, *et al*: 1990, 'A PreciseDetermination of the Number of Families with Light Neutrinos and of the Z Boson Partial Widths', *Phys. Lett. B* **235**, 399.

Anderson, C.D., and S.H. Neddermeyer: 1937, 'Note on the Nature of Cosmic Ray Particles', *Phys. Rev.* **51**, 884.

Babu, K.S., and R.N. Mohapatra: 1988, 'Model for Large Transition Magnetic Moment of the Electron Neutrino', *Phys. Rev. Lett.* **63**, 228.

Barbieri, R., and R.N. Mohapatra: 1988, 'Limit on the Magnetic Moment of he Neutrino from Supernova 1987A Observations', *Phys. Rev. Lett.* **61**, 27.

Barut, A.O.: 1980, 'Magnetic Resonances Between Massive and Massless Spin-1/2 Particles with Magnetic Moments', *J. Math. Phys.* **21**, 568.

Bethe, H.: 1935, 'Ionization Power of a Neutrino with Magnetic Moment', *Proc. Camb. Phil. Soc.* **31**, 108.

Boehm, F., and P. Vogel: 1988, *Physics of Massive Neutrinos,* Cambridge Univ. Press, New York.

Carlson, J.F., and J.R. Oppenheimer: 1932, 'The Impacts of Fast Electrons and Magnetic Neutrons', *Phys. Rev.* **41**, 763.

Case, K.M.: 1957, 'Reformulation of the Majorana Theory of the Neutrino', *Phys. Rev.* **107**, 307.

Dolgov, A.D., and Ya.B. Zeldovich: 1981, 'Cosmology and Elementary Particles', *Rev. Mod. Phys.* **53**, 1.

Elliot, S.R., A.A. Hahn, and M.K. Moe: 1987, 'Direct Evidence for Two-Neutrino Double-Beta Decay in ^{82}Se', *Phys. Rev. Lett.* **59**, 2020.

Feshbach, H., and F. Villars: 1958, 'Elementary Relativistic Wave Mechanics of Spin 0 and Spin $\frac{1}{2}$ Particles', *Rev. Mod. Phys.* **30**, 24.

Friedlander, G., and J. Weneser: 1987, 'Solar Neutrinos: Experimental Approaches', *Science* **235**, 760.

Fukugita, M., and T. Yanagide: 1987, 'Particle Physics Model for the Veloshin-Vysatski-Okun Solution to the Solar-Neutrino Problem', *Phys. Rev. Lett.* **58**, 1807.

Goldhaber, A.S., and M.M. Nieto: 1971, 'How to Catch a Photon and Measure Its Mass', *Phys. Rev. Lett.* **26**, 1390.

Han, D., and Y.S. Kim: 1981, 'Little Group for Photons and Gauge Transformations', *Am. J. Phys.* **49**, 348.

Han, D., Y.S. Kim, and D. Son: 1985, 'Unitary Transformation of Photon Polarization Vectors', *Phys. Rev. D* **31**, 328.

Hegstrom, R.A., J.P Chamberlain, K. Seto, and R.G. Watson: 1988, 'Mapping the Weak Chirality of Atoms', *Am. J. Phys.* **56**, 1086.

Kemmer, N.: 1939, 'The Particle Aspect of Meson Theory', *Proc. Roy. Soc. (London)* **A173**, 91.

Klapdor, H.V. (ed.): 1988, *Neutrinos,* Springer-Verlag, Berlin.

Landau, L.: 1957, 'On the Conservation Laws for Weak Interactions', *Nucl. Phys.* **3**, 127.

Lattimer, J.M., and J. Cooperstein: 1988, 'Limits on the Neutrino Magnetic Moment from SN1987A', *Phys. Rev. Lett.* **61**, 23.

Lederman, L.M.: 1989, 'Observations in Particle Physics from Two-Neutrinos to the Standard Model', *Rev. Mod. Phys.* **61**, 547.

Lee, T.D., and C.N. Yang: 1957, 'Parity Nonconservation and a Two-Component Theory of the Neutrino', *Phys. Rev.* **105**, 1671.

Majorana, E.: 1937, 'Teoria Simmetrica dell'Elettrone e del Positrone', *Nuovo Cimento* **14**, 171.

McLennan, J.A., Jr.: 1957, 'Parity Nonconservation and the Theory of the Neutrino', *Phys. Rev.* **106**, 821.

Perl, M.L., G.S. Abrams, A.M. Boyarski, M. Breidenbach, D.D. Briggs, F. Bulos, W. Chinowsky, J.T. Dakin, G.J. Feldman, C.E. Friedberg, D. Fryberger, G. Goldhaber, G. Hanson, F.B. Heile, B. Jean-Marie, J.A. Kadyk, R.R. Larsen, A.M. Litke, D. Lüke, B.A. Lulu, V. Lüth, D. Lyon, C.C. Morehouse, J.M. Paterson, F.M. Pierre, T.P. Pun, P.A. Rapidis, B. Richter, B. Sadoulet, R.F. Schwitters, W. Tanenbaum, G.H. Trilling, F. Vannucci, J.S. Whitaker, F.C. Winkelmann, and J.E. Wiss, 'Evidence for Anomalous Lepton Production in $e^+ - e^-$ Annihilation', *Phys. Rev. Lett.* **35**, 1489.

Proca, A.L.: 1936a, 'Sur la Théorie Ondulatoire des Electrons Positifs et Négatifs', *J. Phys. et Radium* **7**, 347.

Proca, A.L.: 1936b, 'Sur les Équations Fondamentales des Particules Élémentaires', *Compt. Rend.* **202**, 1490.

Pryce, M.H.L.: 1938, 'On the Neutrino Theory of Light', *Proc. Roy. Soc. (London)* **165A**, 247.

Rosen, S.P.: 1988, 'On Seeing the First Double Beta Decay', *Comments Nucl. Part. Phys.* **18**, 31.

Salam, A.: 1957, 'On Parity Conservation and Neutrino Mass', *Nuovo Cimento* **5**, 299.

Schmiedmayer, J., H. Rouch, and P. Riehs: 1988, 'Measurement of the Electric Polarizability of the Neutron', *Phys. Rev. Lett.* **61**, 1065.

Schwarz, M.: 1989, 'The First High-Energy Neutrino Experiment', *Rev. Mod. Phys.* **61**, 527.

Serpe, J.: 1952, 'Sur la Théorie Abrégée des Particules de Spin 1/2', *Physica* **18**, 295.

Steinberger, J.: 1989, 'Experiments with High-Energy Neutrino Beams', *Rev. Mod. Phys.* **61**, 533.

Street, C., and E. Stevenson: 1937, 'Penetrating Corpuscular Component of the Cosmic Radiation', *Phys. Rev.* **51**, .1005

Weinberg, S.: 1964a, 'Feynman Rules for Any Spin. II. Massless Particles', *Phys. Rev.* **134**, B882.

Weinberg, S.: 1964b, 'Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass', *Phys. Rev.* **135**, B1049.

Weneser, J., and G. Friedlander: 1987, 'Solar Neutrinos: Questions and Hypotheses', *Science* **235**, 755.

Weyl, H.: 1929, 'Elektron und Gravitation. I.', *Z. Phys.* **56**, 330.

Wigner, E.P.: 1939, 'On Unitary Representations of the Lorentz Group', *Ann. Math.* **149**, 40.