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Phonon-fracton anharmonic interactions: The thermal conductivity of amorphous materials

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The anharmonic interaction (third-order elastic Hamiltonian) between fractons (short-length-scale vibrational excitations) and phonons (long-length-scale vibrational excitations) is introduced. The relationship between phonon-fracton scattering rates and the thermal conductivity κ is developed. Two relevant anharmonic lifetimes are calculated: two phonons combining into a single fracton, and one phonon combining with a fracton to produce a fracton. The former is important to the behavior of κ in the "plateau" regime; the latter to κ at temperatures above the plateau regime. The latter is interpreted in terms of phonon-assisted fracton hopping, and gives rise to an extra heat conductance which increases linearly in the temperature T. This behavior appears to be in agreement with recent high-temperature measurements of κ in epoxy resin by de Oliveira and Rosenberg.

I. INTRODUCTION

The thermal conductivity κ of amorphous materials has been shown to be of universal form.[1] The lowest-temperature regime is associated with phonon scattering off two-level systems (TLS),[2] with a thermal conductivity increasing as the square of the temperature. At higher temperatures (typically 8–10 K), the thermal conductivity flattens off into a "plateau" regime, remaining relatively constant for another 5–10 K. The plateau is "followed by a continued slow rise at higher temperatures."[3]

A recent "fracton" model for the behavior in the vicinity of the plateau region was formulated by Alexander *et al.*[4] and later amplified by Orbach and Rosenberg.[5] It suggests that phonons are responsible for the thermal conduction in amorphous materials at low temperatures. The quadratic temperature dependence of the thermal conductivity at the lowest temperatures is thereby associated with phonon scattering off the two-level systems.[2] For higher temperatures, the thermally excited vibrational excitations crossover from phononlike to fractonlike at a temperature $T \approx \hbar\omega_c/k_B$, where ω_c is the crossover frequency separating phonon from fracton excitations.[6] Because the fractons are spatially localized,[7] they cannot contribute to the heat current. In this temperature regime, k_BT is larger than the phonon energies. But only the phonons can carry heat. Hence, one is in the DuLong-Petit regime for the heat-carrying phonons, and for a temperature-independent phonon mean free path, the thermal conductivity is a constant. (We shall investigate the temperature dependence of the phonon mean free path in this temperature regime below.) This explanation for the plateau in the thermal conductivity is consistent with scaling tests[5] where ω_c is altered by virtue of changes in material preparation procedures.

The question arises then, quite naturally: What happens at temperatures above the plateau region? How can the spatially localized fractons contribute to the thermal conductivity? The purpose of this paper is to examine the anharmonic phonon-fracton interaction process. We shall show that through this interaction, phonon-induced fracton hopping can contribute to the heat current, generating a thermal conductivity which increases linearly with increasing temperature. Such behavior has been known approximately,[1] with very recent experiments exhibiting a thermal conductivity which increases linearly with temperature much more clearly.[8]

We shall divide our presentation into sections, with Sec. II the formulation of the three vibrational quanta anharmonic interaction. Sec. III is a calculation of the phonon and fracton lifetimes caused by the anharmonic processes:

$$\text{phonon} + \text{phonon} \leftrightarrow \text{fracton} ,$$

$$\text{phonon} + \text{fracton} \leftrightarrow \text{fracton} .$$

The resultant contribution to the thermal conductivity is developed in Sec. IV. Section V will discuss the experimental situation. We shall show that the measurements of de Oliveira and Rosenberg[8] are consistent with the substance of our calculation. They measure the thermal conductivity of epoxy resin through and above the plateau temperature. They find a contribution to κ linear in temperature above the plateau temperature which can be regarded as an addition to κ at the plateau value. That is, $\kappa_{\text{expt}} = \kappa_{\text{plateau}} + CT$, where C is a constant. This is consistent with the concept of fracton hopping: an additional contribution to the heat current for temperatures above the plateau temperature.

An alternate picture of Karpov and Parshin[9] for the behavior of the thermal conductivity above the plateau temperature introduces anharmonic modes which can carry heat. These states scatter off the two-level systems (TLS) which are common to amorphous structures. At temperatures above the plateau temperature, the relative population of the TLS is inversely proportional to temperature. Thus, the scattering of the anharmonic modes decreases linearly with increasing temperature. Hence, this model ascribes the linear temperature dependence of κ above the plateau temperature to a linear *decrease* in the *scattering* rate of these heat-carrying states. This is to be compared to the fracton-hopping concept introduced in this paper which generates an *additional* heat *transport* mechanism. While it will be difficult to choose between models, it will be argued that the fracton approach has, at the least, the fewest number of arbitrary parameters, and provides a coherent picture of thermal transport over the full temperature range. Our results are summarized in Sec. VI.

II. THE PHONON-FRACTON ANHARMONIC INTERACTION

The conventional anharmonic interaction is written as,

$$\mathscr{H}=(\gamma/V)\int d\mathbf{r}(\nabla\cdot\mathbf{u})^3\ , \tag{1}$$

where γ is the anharmonic interaction constant, V is the volume, and $\mathbf{u}$ the displacement operator. We use the conventional normal-mode expansion for $\mathbf{u}(\mathbf{r})$,

$$\mathbf{u}(\mathbf{r})=\sum_{\alpha}[1/(2\rho\omega_\alpha)]^{1/2}\hat{\mathbf{e}}_\alpha[\phi_\alpha(\mathbf{r})b_\alpha+\phi_\alpha^*(\mathbf{r})b_\alpha^\dagger]\ , \tag{2}$$

where α is the normal-mode index, ρ is the average mass density, $\hat{\mathbf{e}}_\alpha$ is a unit vector in the polarization direction, and $\phi_\alpha(\mathbf{r})$ the normalized vibrational wave function for the αth mode. For ease of notation, we shall drop the unit vector $\hat{\mathbf{e}}_\alpha$ in the subsequent calculations. There are two principal anharmonic diagrams, pictured in Figs. 1(a) and 1(b), representing respectively,

$$\text{phonon} + \text{phonon} \leftrightarrow \text{fracton}\ , \tag{3a}$$

and

$$\text{phonon} + \text{fracton} \leftrightarrow \text{fracton}\ . \tag{3b}$$

The former will be important for the behavior of the thermal conductivity at and above the plateau temperature regime; the latter for the fracton-"hopping" contribution to the thermal conductivity.

The process pictured in Fig. 1(a) [Eq. (3a)] generates a phonon and a fracton lifetime. We shall calculate the formal expressions for both below, and evaluate them explicitly in Sec. III. The process pictured in Fig. 1(b) [Eq. (3b)] will be used to calculate the fracton-hopping rate, and will be evaluated explicitly in Sec. IV.

(a) phonon + phonon ↔ fracton

The interaction Hamiltonian associated with process Eq. (3a) follows from Eqs. (1) and (2),

$$\mathscr{H}=\gamma\sum_{\alpha,\alpha',\alpha''}(A_{\alpha,\alpha'',\alpha'}b_{\alpha'}^\dagger b_{\alpha''}b_\alpha+A_{\alpha,\alpha'',\alpha'}^*b_\alpha^\dagger b_{\alpha''}^\dagger b_{\alpha'})\ , \tag{4}$$

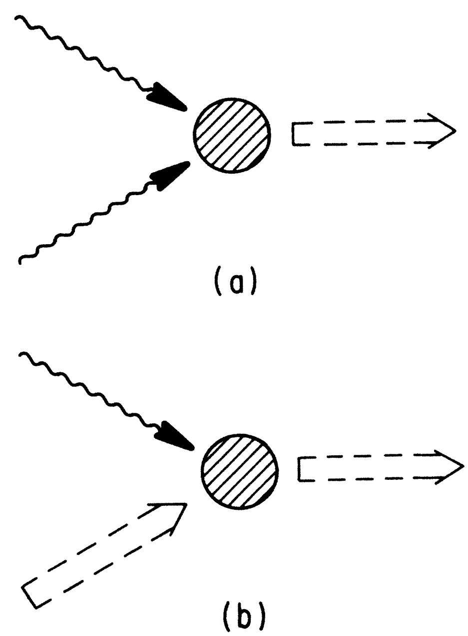

FIG. 1. The decay channels caused by vibrational anharmonicity: (a) phonon + phonon ↔ fracton, and (b) phonon + fracton ↔ fracton. The wavy lines represent the phonons, and the dashed double lines the fractons. The dashed circle represents the interaction Hamiltonian Eq. (1).

where the matrix element $A_{\alpha,\alpha'',\alpha'}$ is

$$A_{\alpha,\alpha'',\alpha'}=[(1/2\rho\omega_\alpha)(1/2\rho\omega_{\alpha''})(1/2\rho\omega_{\alpha'})]^{1/2}\times(1/V)\int d\mathbf{r}[\partial\phi_\alpha(\mathbf{r})/\partial r][\partial\phi_{\alpha''}(\mathbf{r})/\partial r]\times[\partial\phi_{\alpha'}^*(\mathbf{r})/\partial r]\ . \tag{5}$$

The phonon lifetime can be extracted from the transition probability per unit time for the state α to combine with α'' to yield the state α'. Conventional second-order time-dependent perturbation theory generates,

$$W_{\alpha,\alpha''\to\alpha'}=2\pi\gamma^2|A_{\alpha,\alpha'',\alpha'}|^2n_\alpha n_{\alpha''}(1+n_{\alpha'})\times\delta(\omega_{\alpha'}-\omega_\alpha-\omega_{\alpha''})\ . \tag{6}$$

This results in the time rate of change of the occupation of state α,

$$\frac{dn_\alpha}{dt}=2\pi\gamma^2\sum_{\alpha'',\alpha'}|A_{\alpha,\alpha'',\alpha'}|^2[n_{\alpha'}(1+n_{\alpha''})(1+n_\alpha)-n_\alpha n_{\alpha''}(1+n_{\alpha'})]\delta(\omega_\alpha+\omega_{\alpha''}-\omega_{\alpha'})\ . \tag{7}$$

This leads to an expression for the phonon lifetime,

$$(1/\tau_\alpha)_{\text{ph}}^{(a)}=2\pi\gamma^2\sum_{\alpha'',\alpha'}|A_{\alpha,\alpha'',\alpha'}|^2\delta(\omega_{\alpha'}-\omega_\alpha-\omega_{\alpha''})(n_{\alpha''}^0-n_{\alpha'}^0)\ , \tag{8}$$

where n_α^0 is the equilibrium Bose factor $[\exp(\hbar\omega_\alpha/k_BT)-1]^{-1}$. For the remainder of this paper, we shall set $\hbar=1$, for ease of notation.

The fracton lifetime associated with Fig. 1(a) [Eq. (3a)] is obtained in an analogous fashion. The time rate of change of the number of fractons in the state α' is given by,

$$\frac{dn_{\alpha'}}{dt}=2\pi\gamma^2\sum_{\alpha,\alpha''}|A_{\alpha,\alpha'',\alpha'}|^2[(1+n_{\alpha'})n_\alpha n_{\alpha''}-n_{\alpha'}(1+n_\alpha)(1+n_{\alpha''})]\delta(\omega_{\alpha'}-\omega_\alpha-\omega_{\alpha''})\ . \tag{9}$$

This leads to an expression for the fracton lifetime,

$$(1/\tau_{\alpha'})^{(a)}_{\rm fr}=2\pi\gamma^2\sum_{\alpha,\alpha''}|A_{\alpha,\alpha'',\alpha'}|^2\delta(\omega_{\alpha'}-\omega_\alpha-\omega_{\alpha''})(1+n^0_\alpha+n^0_{\alpha''})\ . \tag{10}$$

The matrix element for both lifetimes is found from Eq. (5):

$$\begin{aligned}A_{\alpha,\alpha'',\alpha'}&=-k_\alpha k_{\alpha''}[(1/2M\omega_\alpha)(1/2M\omega_{\alpha''})(1/2M\omega_{\alpha'})]^{1/2}(1/V)^{1/2}\int d\mathbf{r}\exp[i(\mathbf{k}_\alpha+\mathbf{k}_{\alpha''})\cdot\mathbf{r}][\partial\phi^*_{\alpha'}(\mathbf{r})/\partial r]\\&=(i/v_s^2)[(1/2M\omega_{\alpha'})(\omega_\alpha/2M)(\omega_{\alpha''}/2M)]^{1/2}(k_\alpha+k_{\alpha''})(1/V)^{1/2}\int d\mathbf{r}\exp[i(\mathbf{k}_\alpha+\mathbf{k}_{\alpha''})\cdot\mathbf{r}]\phi^*_{\alpha'}(\mathbf{r})\ ,\end{aligned} \tag{11}$$

where v_s is the phonon sound speed. We shall not carry out the detailed angular integrations inherent in Eq. (11) because of their algebraic complexity. Rather, we shall obtain a reasonable estimate of the relaxation rates by using $k_\alpha=\omega_\alpha/v_s$ and $k_{\alpha''}=\omega_{\alpha''}/v_s$. Then, by virtue of the energy conserving delta function, $k_\alpha+k_{\alpha''}\simeq\omega_{\alpha'}/v_s$. Using Eqs. (8) and (10),

$$\begin{aligned}A_{\alpha,\alpha'',\alpha'}&=(i/v_s^3)(\omega_{\alpha'}\omega_\alpha\omega_{\alpha''}/8M^3)^{1/2}(1/V)^{1/2}\\&\quad\times\int d\mathbf{r}\exp(i\mathbf{K}_{\alpha'}\cdot\mathbf{r})\phi^*_{\alpha'}(\mathbf{r}),\ \ \mathbf{K}_{\alpha'}=\mathbf{k}_\alpha+\mathbf{k}_{\alpha''}\ .\end{aligned} \tag{12}$$

The integral in Eq. (12) will turn out to be nontrivial because the phase factor will vary rapidly over the region of integration (i.e., over the volume occupied by the fracton of energy $\omega_{\alpha'}$). We defer its evaluation until Sec. III.

(b) phonon + fracton ↔ fracton

The interaction Hamiltonian associated with process Eq. (3b) follows from Eqs. (1) and (2). We shall not generate the explicit expressions for the phonon and fracton lifetimes associated with this diagram. They follow in a manner similar to Eqs. (8) and (10). We shall evaluate them in the next section.

III. PHONON AND FRACTON LIFETIMES

(a) phonon(α) + phonon(α'') ↔ fracton(α')

Phonon lifetime $(1/\tau_\alpha)^{(a)}_{\rm ph}$

Evaluating the diagram of Fig. 1(a) [Eq. (3a)] from Eq. (8), and using the matrix element for the interaction, Eq. (12), we can find the lifetime of the phonon in the state α. However, the evaluation of matrix element requires careful consideration of the length scales in the problem. We need to evaluate

$$\int d\mathbf{r}\exp[i(\mathbf{k}_\alpha+\mathbf{k}_{\alpha''})\cdot\mathbf{r}]\phi^*_{\alpha'}(\mathbf{r})\ . \tag{13}$$

The form for the fracton wave function, $\phi_{\alpha'}(\mathbf{r})$ has been posited by Alexander *et al*:[10]

$$\phi_{\alpha'}(\mathbf{r})\propto(1/r)^{(d-D)/2}(l_{\omega_{\alpha'}})^{-D/2}\exp[-\tfrac{1}{2}(r/l_{\omega_{\alpha'}})^{d_\phi}]\ . \tag{14}$$

Here, D is the fractal dimensionality,[11] $l_{\omega_{\alpha'}}$ is the characteristic length associated with the localized fracton:[6]

$$l_{\omega_{\alpha'}}\simeq a_0(\omega_{\alpha'}/\Omega_{\rm FD})^{-\bar{\bar{d}}/D}\ , \tag{15}$$

where a_0 is the microscopic length scale, $\Omega_{\rm FD}$ is the fracton Debye frequency,[4] $\bar{\bar{d}}$ is the fracton dimensionality,[6] and d_ϕ is the exponent recognizing the "superlocalization" of the fracton state.[12] In general, d_ϕ is sandwiched between unity and $d_{\min}$, where $d_{\min}$ is defined by $L\propto R^{d_{\min}}$, L being the shortest connected path between two points on the fractal separated by a Pythagorean distance R.[13] For a percolating network in $d=2$, $d_{\min}\simeq1.38$. The localized nature of $\phi_{\alpha'}$ will result in r being no larger than $l_{\omega_{\alpha'}}$ in Eq. (13), so that the exponent will be of magnitude,

$$K_{\alpha'}l_{\omega_{\alpha'}}\sim(\omega_{\alpha'}/v_s)(\omega_{\alpha'})^{-\bar{\bar{d}}/D}\sim(\omega_{\alpha'}/\omega_c)^{\theta/(2+\theta)}>1\ , \tag{16}$$

where we have used the relation[6] $\bar{\bar{d}}=2D/(2+\theta)$, where θ is a measure of the range dependence of the diffusion constant for diffusion on a fractal ($\theta>0$).[14] Here, ω_c is the crossover frequency between phonon and fracton regimes,[6] and we have used the relation $v_s\propto\omega_c^{\theta/(2+\theta)}$.[15] This means that the exponential in (14) will oscillate over the region of integration.

We handle the oscillations as follows. The fracton localization volume is $(l_{\omega_{\alpha'}})^D$. We divide it into regions of volume $(1/K_{\alpha'})^D$, over which the scattering is coherent. The amplitude of the fracton wave function in this volume is proportional to $(l_{\omega_{\alpha'}})^{-D/2}$, and the number of network sites in each region is $(1/K_{\alpha'})^D$. The contribution to the Fourier integral from each region is

$$\sum_\nu[\exp(i\mathbf{K}_{\alpha'}\cdot\mathbf{R}_\nu)](1/l_{\omega_{\alpha'}})^{D/2}\ . \tag{17}$$

where $\mathbf{R}_\nu$ is the position of the νth site in the region of volume $(1/K_{\alpha'})^D$. Equations (8) and (10) require the absolute square of the Fourier integral Eq. (13). Performing the absolute square of Eq. (17) results in

$$(1/K_{\alpha'})^{2D}(1/l_{\omega_{\alpha'}})^D\left|1+\sum_{\substack{\nu'\\ \nu'\neq\nu}}\exp[i\mathbf{K}_{\alpha'}\cdot(\mathbf{R}_\nu-\mathbf{R}_{\nu'})]\right|\sim(1/K_{\alpha'})^{2D}(1/l_{\omega_{\alpha'}})^D\ .$$

Multiplying by the number of such regions, $(l_{\omega_{\alpha'}}K_{\alpha'})^D$, we find for the absolute square of the Fourier integral,

$$(1/K_{\alpha'})^{2D}(1/l_{\omega_{\alpha'}})^{D}(l_{\omega_{\alpha'}}K_{\alpha'})^{D}=(1/K_{\alpha'})^{D}=(v_s/\omega_{\alpha'})^{D} . \tag{18}$$

Inserting Eq. (18) into the absolute square of Eq. (12),

$$|A_{\alpha,\alpha'',\alpha'}|^2 \propto (v_s/a_0)^{D-6}\Omega_{\rm FD}^3\omega_\alpha\omega_{\alpha''}\omega_{\alpha'}^{1-D} . \tag{19}$$

Inserting Eq. (19) into Eq. (8), we obtain the expression for the phonon lifetime for the anharmonic process Eq. (3a):

$$(1/\tau_\alpha)_{\rm ph}^{(a)} \propto \gamma^2(v_s/a_0)^{D-6-d}\Omega_{\rm FD}^{3-\bar{\bar{d}}}\omega_\alpha \times \int_{\omega_c}^{\omega_c+\omega_\alpha} d\omega_{\alpha'}(\omega_{\alpha'}-\omega_\alpha)^{d}\omega_{\alpha'}^{\bar{\bar{d}}-D} \times[n^0(\omega_{\alpha'}-\omega_\alpha)-n^0(\omega_{\alpha'})] , \tag{20}$$

where we have used for the phonon and fracton densities of states,[15]

$$N_{\rm ph}(\omega)=[1/(v_s/a_0)^d]\omega^{d-1} ,$$
$$N_{\rm fr}(\omega)=[1/(\Omega_{\rm FD})^{\bar{\bar{d}}}]\omega^{\bar{\bar{d}}-1} .$$

The evaluation of the integral in Eq. (20) is tedious, but can be shown to equal (for $\omega_\alpha \ll \omega_c$)

$$\begin{array}{ll} k_BT\omega_c^{d+\bar{\bar{d}}-D-2}\omega_\alpha^2, & \beta\omega_c<1 \\ (1/k_BT)\omega_c^{d+\bar{\bar{d}}-D}\omega_\alpha^2\exp(-\beta\omega_c), & \beta\omega_c>1 . \end{array} \tag{21}$$

Inserting this expression into Eq. (20) yields our final result for the phonon lifetime for the process Eq. (3a):

$$(1/\tau_\alpha)_{\rm ph}^{(a)}=\gamma^2(\omega_\alpha^3/\Omega_{\rm FD}^4)(v_s/a_0\Omega_{\rm FD})^{D-d-6}(\omega_c/\Omega_{\rm FD})^{d+\bar{\bar{d}}-D}\times\begin{cases} k_BT\Omega_{\rm FD}/\omega_c^2, & \beta\omega_c<1 \\ \Omega_{\rm FD}\exp(-\beta\omega_c)/k_BT, & \beta\omega_c>1 . \end{cases} \tag{22}$$

This result should be measurable in acoustic attenuation experiments.

Fracton lifetime $(1/\tau_{\alpha'})_{\rm fr}^{(a)}$

An entirely analogous procedure can be followed for the fracton lifetime. Here, the phonon density of states enters twice, so that d in Eq. (20) is replaced by $2d$, and there is no $\bar{\bar{d}}$ left in the exponent. Omitting the algebra, we find,

$$(1/\tau_{\alpha'})_{\rm fr}^{(a)}=(\gamma^2/\Omega_{\rm FD})(v_s/a_0\Omega_{\rm FD})^{D-6-2d}(\omega_{\alpha'}/\Omega_{\rm FD})^{2d-D+1}\times\begin{cases} k_BT/\Omega_{\rm FD}, & \beta\omega_{\alpha'}<1 \\ \omega_{\alpha'}/\Omega_{\rm FD}, & \beta\omega_{\alpha'}>1 . \end{cases} \tag{23}$$

(b) phonon(α) + fracton(α'') $\leftrightarrow$ fracton(α')

Phonon lifetime $(1/\tau_\alpha)_{\rm ph}^{(b)}$

The calculation follows that for process Eq. (3a), with the exception of the matrix element. Simplifying Eq. (5), we have

$$A_{\alpha,\alpha'',\alpha'}=i[(1/2\rho\omega_{\alpha'})(1/2\rho\omega_{\alpha''})(\omega_\alpha/2M)]^{1/2}(1/v_sV)\int d\mathbf{r}\exp(i\mathbf{k}_\alpha\cdot\mathbf{r})[\partial\phi_{\alpha'}(\mathbf{r})/\partial r][\partial\phi_{\alpha''}(\mathbf{r})/\partial r] . \tag{24}$$

Using Eq. (14) for the fracton wave function, we obtain,

$$A_{\alpha,\alpha'',\alpha'}=i[(1/2\rho\omega_{\alpha'})(1/2\rho\omega_{\alpha''})(\omega_\alpha/2M)]^{1/2}(1/v_sVa_0^2)(\omega_{\alpha'}\omega_{\alpha''}/\Omega_{\rm FD}^2)^q[1/l_{\omega_{\alpha'}}l_{\omega_{\alpha''}}]^{D/2} \times\int d\mathbf{r}(1/r)^{d-D}\exp(i\mathbf{k}_\alpha\cdot r)\exp[-\tfrac{1}{2}|(r-R_{\alpha'})/l_{\omega_{\alpha'}}|^{d_\phi}-\tfrac{1}{2}|(r-R_{\alpha''})/l_{\omega_{\alpha''}}|^{d_\phi}] , \tag{25}$$

where $q=\bar{\bar{d}}d_\phi/D$. The integral in Eq. (25) can be approximated by

$$l_{\omega_{\alpha'}}^D\exp[i\mathbf{k}_\alpha\cdot\mathbf{R}_{\alpha'}-\tfrac{1}{2}|(\mathbf{R}_{\alpha'}-\mathbf{R}_{\alpha''})/l_{\omega_{\alpha''}}|^{d_\phi}]+l_{\omega\alpha''}^D\exp[i\mathbf{k}_\alpha\cdot\mathbf{R}_{\alpha''}-\tfrac{1}{2}|(\mathbf{R}_{\alpha'}-\mathbf{R}_{\alpha''})/l_{\omega_{\alpha'}}|^{d_\phi}] .$$

The absolute square of the matrix element, Eq. (24), then becomes

$$|A_{\alpha,\alpha'',\alpha'}|^2=(a_0\Omega_{\rm FD}/v_s)^2(\omega_\alpha/\Omega_{\rm FD})(\omega_{\alpha'}\omega_{\alpha''}/\Omega_{\rm FD}^2)^{2q-1} \times\{(l_{\omega_{\alpha'}}/l_{\omega_{\alpha''}})^D\exp[-|(\mathbf{R}_{\alpha'}-\mathbf{R}_{\alpha''})/l_{\omega_{\alpha''}}|^{d_\phi}]+(l_{\omega_{\alpha''}}/l_{\omega_{\alpha'}})^D\exp[-|(\mathbf{R}_{\alpha'}-\mathbf{R}_{\alpha''})/l_{\omega_{\alpha'}}|^{d_\phi}]\} . \tag{26}$$

Note that Eq. (26) is dimensionless, as it should be. We now insert this expression into a form closely paralleling Eq. (20). Here, however, we can make explicit use of a small phonon energy compared to the fracton energies. That is, $\omega_\alpha \ll \omega_{\alpha'},\omega_{\alpha''}$. We therefore can take $\omega_{\alpha'}\sim\omega_{\alpha''}$ in our subsequent discussion. The phonon relaxation rate for process Eq. (3b) can be found in a manner analogous to Eq. (20). One finds, using Eq. (26) for the absolute square of the matrix element,

$$(1/\tau_\alpha)_{\rm ph}^{(b)}\simeq\gamma^2(a_0\Omega_{\rm FD}/v_s)^2(\omega_\alpha/\Omega_{\rm FD})(\omega_{\alpha'}/\Omega_{\rm FD})^{4q-2}(\beta\omega_\alpha/\delta)(n_{\alpha'}^0+1)n_{\alpha'}^0\exp[-|(\mathbf{R}_{\alpha'}-\mathbf{R}_{\alpha''})/l_{\omega_{\alpha'}}|^{d_\phi}] . \tag{27}$$

The factor δ represents the sum of the energy widths of the fracton states. This quantity enters the "golden-rule" equation because of the localized character of the fracton wave functions.[7] An extensive discussion of the role played by δ can be found in Ref. 10.

The result, Eq. (27) must be averaged over the probability densities for finding fractons of energy $\omega_{\alpha'} \simeq \omega_{\alpha''}$ at positions $\mathbf{R}_{\alpha'}$ and $\mathbf{R}_{\alpha''}$, respectively. The probability density for finding the first (α') fracton at $\mathbf{R}_{\alpha'}, \mathbf{R}_{\alpha'}+d\mathbf{R}_{\alpha'}$ with energy in the interval, $\omega_{\alpha'}, \omega_{\alpha'}+d\omega_{\alpha'}$, is[10] $DR_{\alpha'}^{D-1}dR_{\alpha'}N_{\text{fr}}(\omega_{\alpha'})d\omega_{\alpha'}$. The probability density for finding the second (α'') fracton at $\mathbf{R}_{\alpha''}$, $\mathbf{R}_{\alpha''}+d\mathbf{R}_{\alpha''}$ with energy in the interval $\omega_{\alpha''}$, $\omega_{\alpha''}+\delta$, is[10] $DR_{\alpha''}^{D-1}dR_{\alpha''}N_{\text{fr}}(\omega_{\alpha''})\delta$. We transform the integrations over $\mathbf{R}_{\alpha'}$ and $\mathbf{R}_{\alpha''}$ into an integration over the difference in position of the two fractons, $R = |\mathbf{R}_{\alpha'} - \mathbf{R}_{\alpha''}|$. Equation (27) becomes

$$(1/\tau_\alpha)_{\text{ph}}^{(\text{b})} \simeq \gamma^2 (a_0 \Omega_{\text{FD}}/v_s)^2 (\beta\omega_\alpha^2/\Omega_{\text{FD}}) \int DR^{D-1} dR N_{\text{fr}}^2(\omega_{\alpha'}) d\omega_{\alpha'} (\omega_{\alpha'}/\Omega_{\text{FD}})^{4q-2} \times \exp(\beta\omega_{\alpha'})[\exp(\beta\omega_{\alpha'})-1]^{-2} \exp[-(R/l_{\omega_{\alpha'}})^{d_\phi}] . \quad (28)$$

The R integration yields $(l_{\omega_{\alpha'}}/a_0)^D \Gamma[(D/d_\phi)+1]$. Performing the $\omega_{\alpha'}$ integration yields

$$(1/\tau_\alpha)_{\text{ph}}^{(\text{b})} \simeq \gamma^2 (a_0 \Omega_{\text{FD}}/v_s)^2 (\beta\omega_\alpha^2/\Omega_{\text{FD}}^3)(\omega_c/\Omega_{\text{FD}})^{4q-4+\bar{\bar{d}}} k_B T \times \begin{cases} k_B T/\omega_c, & \beta\omega_c < 1 \\ \exp(-\beta\omega_c), & \beta\omega_c > 1 . \end{cases} \quad (29)$$

Fracton lifetime $(1/\tau_{\alpha''})_{\text{fr}}^{(\text{b})}$

This process has been referred to in the Introduction as phonon-assisted fracton hopping. For the evaluation of the rate, we make use of the absolute square of the matrix element, Eq. (26), under the conditions that $\omega_{\alpha'} \sim \omega_{\alpha''}$. However, we now must sum over the final positions of the fracton, $\mathbf{R}_{\alpha'}$. We do this by requiring that there be a probability unity to find the final-state fracton at a distance $R(\omega_{\alpha''})$ away from the initial fracton at $\mathbf{R}_{\alpha''}$. This distance is given by the relation,

$$N_{\text{fr}}(\omega_{\alpha''})\omega_c[R(\omega_{\alpha''})]^D \sim 1 . \quad (30)$$

Here, ω_c reflects the maximum phonon energy, and we are assuming that the fracton "hop" distance $R(\omega_{\alpha''}) < \xi$, the length scale associated with fracton-phonon crossover. For fracton hopping over larger length scales, the exponent D in Eq. (30) is replaced by the Euclidean dimension d. We shall exhibit results for the latter condition below Eq. (32). Using our expression for $l_{\omega_{\alpha''}}$ from Eq. (15), Eq. (30) results in the ratio

$$R(\omega_{\alpha''})/l_{\omega_{\alpha''}} \sim (\omega_{\alpha''}/\omega_c)^{1/D} > 1 . \quad (31)$$

We see, therefore, that the fracton is "hopping" a distance greater than the fracton localization distance. This distance will enter into our expression for the thermal conductivity in Sec. IV. Using the interaction Hamiltonian Eq. (4), we obtain

$$\{1/\tau_{\alpha''}[R(\omega_{\alpha''})<\xi]\}_{\text{fr}}^{(\text{b})} \simeq \gamma^2 (1/2Mv_s^2)(a_0/v_s)^d \omega_c^{d-1} k_B T (\omega_{\alpha''}/\Omega_{\text{FD}})^{4q-2} \exp[-(\omega_{\alpha''}/\omega_c)^{d_\phi/D}] , \quad (32)$$

using Eq. (31). When the reverse is true, $R(\omega_{\alpha''}) > \xi$, the mean hopping distance is given by [see Eq. (30) for the opposite limit],

$$N_{\text{fr}}(\omega_{\alpha''})\omega_c[R(\omega_{\alpha''})]^d \sim 1 . \quad (33)$$

Again, the ratio $R(\omega_{\alpha''})/l_{\omega_{\alpha''}} > 1$ [upon replacing D by d in Eq. (31)]. This results in

$$\{1/\tau_{\alpha''}[R(\omega_{\alpha''})>\xi]\}_{\text{fr}}^{(\text{b})} \simeq \gamma^2 (1/2Mv_s^2)(a_0/v_s)^d \omega_c^{d-1} k_B T (\omega_{\alpha''}/\Omega_{\text{FD}})^{4q-2} \exp[-(\omega_{\alpha''}/\omega_c)^{d_\phi/d}] . \quad (34)$$

This completes our calculation of the phonon and fracton lifetimes according to the anharmonic decay processes exhibited in Fig. 1 and Eq. (3). We shall now use Eqs. (32) and (34) to calculate the *extra* thermal conductivity brought about by phonon-assisted fracton hopping.

IV. FRACTON-HOPPING CONTRIBUTION TO THE THERMAL CONDUCTIVITY

We have derived the phonon-assisted fracton-hopping rates in Sec. III. We shall use them to construct the diffusion constant for heat in this section, and proceed to calculate the concomitant contribution to the thermal conduction. The thermal conductivity κ is given by

$$\kappa = \int d\omega_{\alpha''} N_{\text{fr}}(\omega_{\alpha''}) C(\omega_{\alpha''}) D(\omega_{\alpha''}) , \quad (35)$$

where $N_{\text{fr}}(\omega_{\alpha''})$ is the fracton energy density of states,[4,6] $C(\omega_{\alpha''})$ the specific heat of the α'' mode, and $D(\omega_{\alpha''})$ the diffusion constant for fracton hopping.

We start with the diffusion constant. For fracton hopping,

$$D_{\alpha''}(\omega_{\alpha''}) \simeq (1/\tau_{\alpha''}) R^2(\omega_{\alpha''}) , \quad (36)$$

where $1/\tau_{\alpha''}$ is the phonon-assisted fracton-hopping rate [Eqs. (32) and (34)], and $R(\omega_{\alpha''})$ the mean hopping distance [Eqs. (30) and (33), respectively]. We can rewrite Eq. (36) for the two limits $R(\omega_{\alpha''}) \lessgtr \xi$,

$$[D_{\alpha''}(\omega_{\alpha''})]_{R(\omega_{\alpha''})<\xi}=\{1/\tau_{\alpha''}[R(\omega_{\alpha''})<\xi]\}\times l^2_{\omega_{\alpha''}}(\omega_{\alpha''}/\omega_c)^{2/D}\ , \quad (37a)$$

$$[D_{\alpha''}(\omega_{\alpha''})]_{R(\omega_{\alpha''})>\xi}=\{1/\tau_{\alpha''}[R(\omega_{\alpha''})>\xi]\}\times l^2_{\omega_{\alpha''}}(\omega_{\alpha''}/\omega_c)^{2/d}\ . \quad (37b)$$

The specific heat carried by the α'' mode is,

$$C(\omega_{\alpha''})=\omega_{\alpha''}(\partial n^0_{\alpha''}/\partial T)\ . \quad (38)$$

It will turn out that the principal contribution to the thermal conductance will arise from fractons near the phonon-fracton crossover energy ω_c. Because ω_c/k_B is of the order of the plateau temperatures, and we are interested primarily in temperatures well above the plateau temperatures, $C(\omega_{\alpha''})\sim k_B$. Inserting this value into Eq. (35), and then using Eq. (37a), we obtain the thermal conductivity arising from fracton hopping:

$$\kappa\,|_{R(\omega_{\alpha''})<\xi}=\int_{\omega_c}^{k_BT} d\omega_{\alpha''}N_{\rm fr}(\omega_{\alpha''})k_B\gamma^2(k_BT/2Mv_s^2)[\omega_c^{d-1}/(v_s/a_0)^d]a_0^2(\omega_{\alpha''}/\Omega_{\rm FD})^{4q-2}\times(\omega_{\alpha''}/\omega_c)^{2/D}(\omega_{\alpha''}/\Omega_{\rm FD})^{-4/(2+\theta)}\exp[-(\omega_{\alpha''}/\omega_c)^{d_\phi/D}]\ . \quad (39)$$

using Eq. (15). We obtain

$$\kappa\,|_{R(\omega_{\alpha''})<\xi}=\gamma^2k_B^2T(1/2M\Omega_{\rm FD}^4)(\omega_c/\Omega_{\rm FD})^{s_1+\bar{\bar{d}}-1}\int_{\omega_c}^{k_BT} d\omega_{\alpha''}(\omega_{\alpha''}/\omega_c)^{s_0+\bar{\bar{d}}-1}\exp[-(\omega_{\alpha''}/\omega_c)^{d_\phi/D}]\ , \quad (40)$$

where

$$s_1=4q-2-[4/(2+\theta)]+d-1-[\theta(d+2)/(2+\theta)] \quad (41a)$$

and

$$s_0=4q-2-[4/(2+\theta)]+(2/D)\ . \quad (41b)$$

For $\beta\omega_c<1$, the integral is a number times ω_c:

$$\omega_c\int_1^{1/\beta\omega_c}dx\,x^{s_0+\bar{\bar{d}}-1}\exp(-x^{d_\phi/D})\simeq\omega_c(D/d_\phi)\Gamma[(D/d_\phi)(s_0+\bar{\bar{d}})]\ .$$

Inserting into Eq. (40), we obtain our final result,

$$\kappa\,|_{R(\omega_{\alpha''})<\xi}=(\gamma^2/2M\Omega_{\rm FD}^3)(\omega_c/\Omega_{\rm FD})^{s_1+\bar{\bar{d}}}\times(D/d_\phi)\Gamma[(D/d_\phi)(s_0+\bar{\bar{d}})]k_B^2T\ . \quad (42)$$

This important result shows that the phonon-assisted fracton-hopping contribution to the thermal conductivity is linear in temperature. The origin of the linear temperature dependence of κ as exhibited in Eq. (42) lies with the dispersion law for fractons, Eq. (15). The superlocalization for fracton excitations means that the greatest contribution to the thermal conductivity will arise from those fractons which hop the greatest distance for a given fracton-fracton overlap. This occurs for the lowest-energy fractons, i.e., at the phonon fracton crossover frequency ω_c. But by supposition, $\hbar\omega_c \ll k_BT$ for temperatures above the plateau temperature. Hence, the fracton states which contribute most to the heat transport are those which lie lowest in energy, and therefore have energies much less than k_BT. This means the integral in Eq. (40) is dominated by its lower limit, and is therefore independent of temperature. This results in the simple proportionality of κ to T, independent of the high-energy form for the fracton density of states.

Because the phonon-assisted fracton is an additional contribution to the thermal conductance, it should "add" to the value of the thermal conductivity at the plateau, on the assumption that the phonon mean free path does not change. This condition will depend on the size of the terms calculated in Sec. III, and we shall return to this question later. Given this condition, one would expect to find experimentally that the high-temperature thermal conductivity could be written in the form

$$\kappa_{\rm expt}=\kappa_{\rm plateau}+CT\ , \quad (43)$$

where C is a constant for a particular material, given by Eq. (42). The onset of the linear term in temperature will be more gradual than predicted by Eq. (43) because of the smooth nature of the phonon-fracton crossover.[16]

The opposite limit (the fracton hopping length $>\xi$) can be found using Eq. (37b) in Eq. (35). We find

$$\kappa\,|_{R(\omega_{\alpha''})>\xi}=(\gamma^2/2M\Omega_{\rm FD}^3)(\omega_c/\Omega_{\rm FD})^{s_1+\bar{\bar{d}}}(d/d_\phi)\{1+[2/(2+\theta)](d-D)\}^{-1}\times\Gamma[(d/d_\phi\{1+[2/(2+\theta)](d-D)\})(\tilde{s}_0+\bar{\bar{d}})]k_B^2T\ , \quad (44)$$

where

$$\tilde{s}_0=4q-2+(2/d)-(2\bar{\bar{d}}/d)\ . \quad (45)$$

Apart from numerical factors, this expression for κ has the same functional form as for the opposite limit, Eq. (42).

This completes our calculation of the fracton-hopping contribution to the thermal conduction of fractal networks. We discuss the experimental situation in the next section (Sec. V), and another model for the observed behavior of the thermal conductivity above the plateau temperature.

V. COMPARISON WITH EXPERIMENT AND AN ALTERNATE THEORETICAL MODEL

Radiation corrections to measurements of the thermal conductivity at temperatures above the plateau temperature make the extraction of κ difficult in this temperature range. Our theoretical model (see Sec. IV) suggests that one should observe an *additional* contribution to κ which should simply "add on" to the plateau value of κ. Assuming that the low-frequency ($\omega<\omega_c$) phonon mean free path remains constant at higher temperatures, one would expect the extrapolation of κ back to $T=0$ to strike the ordinate at the plateau value of κ, κ_{plateau}. That is,

$$\kappa=\kappa_{\text{plateau}}+CT,\quad T>T_{\text{plateau}}\ , \tag{46}$$

where C is a constant, in the temperature regime above the plateau temperature. However, the low-frequency phonon mean free path may decrease with increasing temperature, as we have shown in Sec. III. It is difficult to assess the strength of this decrease in the absence of explicit measurements. One such test is a decrease in κ within the plateau temperatures (i.e., the plateau has negative slope). There are examples of such behavior in the literature.[17] If one assumes that the decrease of the low-frequency phonon mean free path with increasing temperature is sufficiently strong, then at temperatures well above the plateau temperature, the form for κ should be

$$\kappa=CT\ . \tag{47}$$

That is, the extrapolation of the high-temperature thermal conductivity back to $T=0$ should pass through the origin.

We exhibit in Fig. 2 some very recent measurements of the thermal conductivity of epoxy resin in the temperature range of interest to this paper. There are a number of features of these measurements which deserve comment. First, the linear temperature dependence of κ above the plateau temperature is quite clear. Second, it will be noted that plateau is relatively "flat," suggesting that the low-frequency phonon mean free path might be expected to remain relatively constant up to the highest temperature of measurement ($T\simeq 90$ K). The solid line in Fig. 2 is the extrapolation of κ back to $T=0$. One sees, remarkably, that the line intersects the ordinate at precisely the value κ_{plateau}, in agreement with Eq. (46). Third, the "onset" of the contribution to κ which is linear in temperature "turns on" much slower than a simple addition of CT to κ_{plateau} would suggest. Instead, there is a gradual transition for κ between a constant value, κ_{plateau}, and $\kappa_{\text{plateau}}+CT$. We believe this is caused by the smooth crossover of the vibrational excitations in epoxy resin from phonon to fracton character. This smooth change has been anticipated in a scaling approach to the crossover problem (see Fig. 1 of Ref. 16).

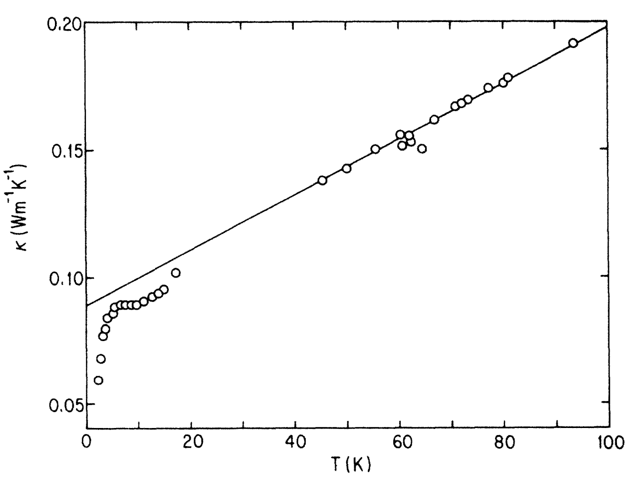

FIG. 2. The thermal conductivity (κ) of epoxy resin, measured as a function of temperature by J. E. de Oliveira and H. M. Rosenberg (private communication). The circles represent the experimental points; the solid line is drawn to show that the high-temperature values of κ extrapolate to the value of κ at the plateau, κ_{plateau}, at $T=0$.

The results exhibited in Fig. 2 have not been available in the literature, so it is difficult to know their relevance to another model for the linear increase of κ above the plateau temperature. Karpov and Parshin[9] take into account the anharmonicity of the local atomic potentials in amorphous materials. They find a logarithmic divergent van Hove singularity in the TLS density of states. The energy at which the singularity occurs is $w\sim 30$ K (not far from the crossover energy we have previously introduced[4] for phonon-fracton crossover). The observed[18–20] peak in the reduced specific heat, $C(T)/T^3$, is ascribed to this singularity. Their calculation shows that the contribution of phonons with energies $\hbar\omega\gg w$ to the thermal conductivity, which stems from the resonant scattering of these phonons by quasilocal harmonic modes, is independent of the temperature at $k_BT\gg w$. The anharmonic modes with energies much less than w make a contribution to the thermal conductivity at $k_BT\gg w$ which increases linearly with temperature. The peaks in the energy density of states (caused by the van Hove singularities) should give rise to a plateau in the thermal conductivity at $k_BT\sim w/3$ because of resonant scattering of phonons in these states.

Their model has some striking similarities to our own. The singularity which they attribute to a van Hove singularity occurs in the same energy region where we expect the crossover to occur between phonon and fracton excitations. The crossover is accompanied by a rather smooth peak in the vibrational density of states.[16] The linear temperature dependence for κ arises in their model from vibrational states with energies less than w. The dispersion relation for fractons[4] causes the majority of the contribution to κ to arise from fractons close to the crossover frequency ω_c. They attribute the plateau to a resonant scattering of phonons. The smooth crossover between fractons and phonons results in precisely the same phenomenon.[21] One might be tempted to argue that their formulation is simply an explicit representation of fracton dynamics. However, the origin of the linear temperature dependence for κ above the plateau temperature arises from very different processes in the two models. For the model of Karpov and Parshin, the anharmonic modes with energies less than w can carry heat. They are scattered by the two-level systems (TLS). The linear temperature increase of κ is ascribed by a *decreased scattering* of the quasilocalized states by the TLS with increasing temperature for $k_B T \gg w$ because of the reduction in relative population of the TLS. The fracton-hopping formulation introduces an *additional heat-carrying channel,* thereby generating an increase in the thermal conductivity with increasing temperature. Thus, the increase of κ with temperature above the plateau is argued to arise from an additional contribution to the heat current, not from a reduction in the scattering cross section for the heat-carrying modes. One will have to wait for detailed comparisons with experiment to determine which model is more relevant. We recognize that our own stake in the fracton-hopping concept may make us less than objective. However, the three features of the experiments detailed at the beginning of this section which appear to be described so satisfactorily by our model do give us some confidence that the fracton-hopping model may be relevant to the thermal transport properties of amorphous materials.

VI. SUMMARY AND CONCLUSIONS

We have introduced the third-order anharmonic interaction between phonons and fractons using the conventional form for vibrational anharmonicity. We have developed expressions for phonon and fracton lifetimes resulting from this interaction. Superlocalization[12] of the fracton states suggests that they should not contribute to thermal conduction. We have shown how the anharmonic interaction can give rise to an additional contribution to the thermal conductivity from phonon-assisted fracton hopping. This process is somewhat analogous to Mott's "variable-range-hopping" conduction mechanism for electrons, though in the fracton case there is no variability in the hopping distance. The fracton dispersion law, in conjunction with superlocalization, results in a linear temperature dependence for the thermal conductivity for temperatures above the crossover energy ($T > \omega_c / k_B$). This prediction agrees well with known experimental results for amorphous materials.

The approach of Karpov and Parshin to the high-temperature thermal conductivity of amorphous materials was outlined. It was shown that there was a fundamental difference between their calculation and our own. They attributed the linear temperature dependence of κ above the plateau temperature to a decrease in scattering of anharmonic modes off the TLS. We instead attribute the linear temperature dependence of κ to an additional heat-conduction channel which becomes important when fracton states are thermally occupied. Which is the more relevant model must await detailed comparison with experiment. For the moment, the phonon-assisted fracton-hopping model does seem to be consistent with all known experimental data for the thermal conductivity of amorphous materials above the plateau temperature.

Note added in proof. After submission of our manuscript, the full paper of V. G. Karpov and D. A. Parshin appeared [Sov. Phys.—JETP **61**, 1308 (1985)], following on their previous letter (our Ref. 9). Their argument for the linear increase in κ above the plateau temperature is centered on the contribution to κ of phonons of low energy, κ_1, ($\omega_1 < \omega/\hbar$ in their notation). They state: "The origin of the $\kappa_1 \propto T$ relation lies in the fact that prethermal ($\hbar\omega \ll k_B T$) phonons are scattered by TLS with level population differences which decrease with increasing T proportionally with T^{-1}. The mean free path then grows linearly with increasing T. . . and their contribution to the thermal conductivity varies in the same way."

The problem with this origin for κ above the plateau temperature is that it appears inconsistent with data for ultrasonic attenuation in glasses [S. Hunklinger and W. Arnold, in *Physical Acoustics,* edited by W. P. Mason and R. N. Thurston (Academic, New York, 1976), Vol. 12, p. 155; J. T. Krauss and C. R. Kurkjian, J. Am. Ceram. Soc. **51**, 226 (1968); and C. K. Jones, P. G. Klemens, and J. A. Rayne, Phys. Lett. **8**, 31 (1964)]. The acoustic attenuation was found to increase monotonically with increasing tempreature, with a region of lesser slope (but always positive) between roughly 5—20 K. The mechanism of Karpov and Parshin would require that the acoustic attenuation should *diminish* with increasing temperature for temperatures above the plateau temperature. The highest frequency used in the experiments of Jones *et al.* was 930 Mc/s (or an equivalent temperature of roughly 0.05 K). It is conceivable that acoustic attenuation experiments at frequencies an order of magnitude higher (of the order of the plateau energy) would exhibit the behavior required by Karpov and Parshin, but the trend of the acoustic attenuation experiments (Jones *et al.*) is in the opposite direction. Increasing frequency leads to a larger *increase* of attenuation with increasing temperature.

We argue, therefore, that there are experimental observations which are unfavorable for the Karpov and Parshin mechanism for the linear increase of κ above the plateau temperature, while, so far at least, our proposal (fracton hopping) appears consistent with all known experimental results.

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[1]Reviewed in *Amorphous Solids: Low Temperature Properties,* edited by W. A. Phillips (Springer, Berlin, 1981).

[2]W. A. Phillips, J. Low Temp. Phys. **7**, 351 (1972); P. W. Anderson, B. I. Halperin, and C. M. Varma, Philos. Mag. **25**, 1

(1972).

[3]J. J. De Yoreo, R. O. Pohl, and Gerald Burns, Phys. Rev. B **32**, 5780 (1985).

[4]S. Alexander, C. Laermans, R. Orbach, and H. M. Rosenberg, Phys. Rev. B **28**, 4615 (1983).

[5]R. Orbach and H. M. Rosenberg, *Proceedings of the 17th International Conference on Low Temperature Physics*, edited by U. Eckern, A. Schmid, W. Weber, and H. Wühl (Elsevier, Amsterdam, 1984), p. 375.

[6]S. Alexander and R. Orbach, J. Phys. (Paris) Lett **43**, L625 (1982).

[7]R. Rammal and G. Toulouse, J. Phys. (Paris) Lett. **44**, L13 (1983).

[8]J. E. de Oliveira and H. M. Rosenberg (private communication).

[9]V. G. Karpov and D. A. Parshin, Pis'ma Zh. Eksp. Teor. Fiz. **38**, 536 (1983) [JETP Lett. **38**, 648 (1983)].

[10]S. Alexander, Ora Entin-Wohlman, and R. Orbach, Phys. Rev. B **32**, 6447 (1985).

[11]B. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, New York, 1983); Ann. Isr. Phys. Soc. **5**, 59 (1983).

[12]Y.-E. Levy and B. Souillard (unpublished).

[13]K. M. Middlemiss, S. G. Whittington, and D. S. Gaunt, J. Phys. A **13**, 1835 (1980); R. Pike and H. E. Stanley, *ibid.* **14**, L169 (1981); D. C. Hong and H. E. Stanley, *ibid.* **16**, L475 (1983); **16**, L525 (1983); H. J. Hermann, D. C. Hong, and H. E. Stanley, *ibid.* **17**, L261 (1984); J. Vannimenus, J. P. Nodal, and C. Martin, *ibid.* **17**, L351 (1984); S. Havlin and R. Nossal, *ibid.* **17**, L427 (1984); S. Havlin, Z. V. Djordjevic, I. Majid, H. E. Stanley, and G. H. Weiss, Phys. Rev. Lett. **53**, 178 (1984).

[14]Y. Gefen, A. Aharony, and S. Alexander, Phys. Rev. Lett. **50**, 77 (1983).

[15]S. Alexander, Ann. Isr. Phys. Soc. **5**, 144 (1983).

[16]A. Aharony, S. Alexander, O. Entin-Wohlman, and R. Orbach, Phys. Rev. B **31**, 2565 (1985).

[17]R. O. Pohl (private communication).

[18]R. C. Zeller and R. O. Pohl, Phys. Rev. B **4**, 2029 (1971).

[19]M. P. Zeitlin and A. C. Anderson, Phys. Rev. B **12**, 4475 (1975).

[20]S. Kelham and H. M. Rosenberg, J. Phys. C **14**, 1737 (1981); C. I. Nichols and H. M. Rosenberg, *ibid.* **17**, 1165 (1984); D. E. Farrell, J. E. de Oliveira, and H. M. Rosenberg, in *Phonon Scattering in Condensed Matter,* edited by W. Eisenmenger *et al.* (Springer-Verlag, Heidelberg, 1984), p. 422.

[21]P. F. Tua, S. J. Putterman, and R. Orbach, Phys. Lett. **98A**, 357 (1983).