International Journal of Modern Physics B, Vol. 8, Nos. 11 & 12 (1994) 1577–1624

MICROSCOPIC MODEL OF SUPERFLUID HELIUM-4

G. C. MARQUES
Instituto de Física da Universidade de São Paulo, C. P. 20516, 01452-990 São Paulo, Brasil
and
Physics Dept., Texas A&M University, College Station, Texas 77843-4242

Received 12 April 1993
Revised 25 October 1993

To understand the properties of superfluid Helium-4 we propose a microscopic model for the description of this imperfect boson gas at very low temperatures. This microscopic theory contains only two parameters that are inferred from the He–He interaction potential. The understanding of the properties of Helium-4 in the super-fluid phase is based on Bose–Einstein condensation and for its description we use field theory at finite temperatures. We get a large number of characteristic features of superfluidity. Among these features we get phonon and roton spectrum, the two-fluid component description and London's relation. We have also made predictions for twelve physically relevant quantities for He-4 below the λ-point. The predictions of the model were made by using the loop expansion in field theory. We find that our predictions are in reasonable agreement with the experimental results.

1. Introduction

One of the distinctive features of Helium-4 is that it remains liquid for temperatures very close to zero (that is, Helium-4 is a quantum liquid). Furthermore, while liquid, Helium-4 exhibits two distinct phases. In phase I it behaves as a normal liquid whereas in phase II it behaves as a superfluid. Helium-4 undergoes a phase transition from phase I to phase II at a temperature $T_\lambda = 2.18°$ K (λ-point).

The quantum liquid behavior of ^{4}He can be understood from the fact that the He–He interaction potential is shallow and has a short range. A dilute system of particles whose two-body interaction has these properties (more precisely, the interparticle potential has short range and exhibits no two-particle bound state) is defined as an imperfect gas. An imperfect gas provides an example of a quantum liquid. The phase transition occurring at the λ-point is understood as due to a macroscopic occupation of the $\mathbf{p} = 0$ state (Bose–Einstein condensation).[1–3]

In order to achieve a description of superfluid Helium-4 and the determination of many physically relevant quantities we shall study in this paper Bose–Einstein condensation in an imperfect gas. What makes possible the understanding of a large number of properties of Helium-4 is phase II is the fact that the description of an imperfect gas at low temperature is roughly insensitive to the details of the interatomic forces. The system is sensitive only to some general features of the potential and, consequently, the complete description of an imperfect gas could be achieved by employing just three parameters.[4,5] Two of them are dynamical parameters and the other one is a kinematical parameter.

The imperfect gas problem can be formulated as the problem of achieving a description of the system at extremely low temperature in terms of these (three) parameters. This would ultimately require the construction of an effective Hamiltonian that contains the essential features of the exact Hamiltonian and that allows for a systematic method for the calculation of the partition function of the system. The effective theory should not contain free parameters since if one is able to infer from the interaction potential its relevant features, one should also be able to infer from it the relevant physical constants.

What are the details of the interatomic potential which are relevant in order to understand an imperfect boson gas at low temperature? In other words, to what features of the potential would the system be sensitive in this limit? We argue that the system is sensitive to a global aspect of the potential (the scattering length) and to a local aspect of the potential (the depth of the potential).

The first dynamical parameter is the scattering length a defined by

$$a = \frac{m}{4\pi}\int V(\mathbf{r})d^3\mathbf{r} \tag{1.1}$$

where V is the two-body interaction potential of the atoms.

From expression (1.1) one can see that the scattering length is just the scattering amplitude in Born's approximation in the zero-momentum limit. In this limit the scattering amplitude is not sensitive to the local aspects of the potential but rather to a global aspect of it. The total cross-section at low energies assumes the form of a geometrical cross section, $\sigma = 4\pi(a)^2$. In this way a is, roughly speaking, the effective diameter of the potential. The scattering looks like that from a hard sphere of diameter a.

The choice of the depth of the potential as another relevant parameter for the description of the system at very low temperatures can be understood within the single particle context. For a particle under the action of an external force, then, at low temperatures, the particle tends to seat at the most stable configuration; that is, the minimum of the potential. Within this classical reasoning the only relevant region of the potential at low temperature would be the minimum of the potential. Furthermore, within the semiclassical approximation the energy will be given, in the zero-temperature limit, by the depth of the potential.

The kinematical parameter is the helium atom mass m.

In the case of He^4 the scattering length a and the depth of the potential ΔE can be read from the interatomic potential. The approximate values for these parameters for Lennard–Jones potential are[6]

$$\begin{aligned} m &\cong 3.5 \cdot 10^9\,\mathrm{eV} \\ \Delta E &\cong 8.81 \cdot 10^{-4}\,\mathrm{eV} = 10.2^\circ\,\mathrm{K} \\ a^{-1} &\cong 0.77 \cdot 10^3\,\mathrm{eV} = (2.6\,\text{Å})^{-1} \end{aligned} \tag{1.2}$$

where in (1.2), as well as throughout our paper, we are using the natural system of units ($k = 1$, $c = 1$, $\hbar = 1$). All relevant results will also be expressed in terms of more conventional units.

Although these parameters differ from each other by at least 6 orders of magnitude (in natural units) they give, as we will show in this paper, a fairly good account of all relevant physical quantities in Helium superfluid phase. The approach used by us makes use of Bose–Einstein condensation[2,7,8] in order to derive all the relevant features of the superfluidity phenomena.

This paper is organized as follows. In Sec. 2 we present some basic features of the superfluidity phenomenon. Throughout this paper we have been concerned with how the microscopic model, here proposed, can account for these characteristic features of superfluidity. In Sec. 3 we describe the framework for dealing with Bose–Einstein condensation. The method employed by us makes use of a field theoretical approach fully developed in Ref. 7. The most relevant result in this section is the conclusion that in the Bose–Einstein condensed phase the Gibbs energy is zero. This result is known to be valid for a noninteracting gas. From this result London's relation and a zero temperature equation of state follows.

In Sec. 4 we construct the effective Hamiltonian of the theory and present the loop expansion in Field theory. The calculation of the relevant thermodynamic potential at the $O + 1$ loop level is presented in Sec. 5. In this section we discuss also a simple approach to symmetry restoration for spontaneously broken symmetry.

The description of the superfluid at very low temperatures is present in Sec. 6. In this section we give an account of the following features of superfluid Helium-4: phonon spectrum, low temperature thermodynamics, ground state energy, Helium density, zero-temperature equation of state, binding energy per particle, sound speed, London relation, roton spectrum and critical velocity.

The two fluid description is presented in Sec. 7. We have been concerned with the identification of the normal component and in presenting an alternative method for making calculations of thermodynamical variables for the normal and superfluid components.

In Sec. 8 we have been mainly concerned with simple estimates of the critical temperature. We have made simple calculations and, by making the gap equal too zero, we have to got a good estimate for T_λ.

We end this paper with Sec. 9 which is devoted to conclusions. The main results of this paper are summarized into two tables. In Table 2 we present 12 specific numerical predictions. In Table 1 we present essential features of the superfluidity phenomenon.

2. Liquid Helium at Low Temperature[9–13]

2.1. *Excitation spectrum*

Liquid Helium-4 exhibits two distinct phases. In phase I Helium-4 behaves as a normal liquid whereas in phase II it exhibits superfluidity. Helium-4 undergoes a phase transition from phase I to phase II at a temperature $T_\lambda = 2.18^\circ$ K (λ-point). In phase II one can distinguish two regimes. For temperatures in the range 1.0° K $< T <$ 2.18° K the liquid behaves as a mixture of a "normal component" and a "superfluid component" (anomalous region). Below 1° K the normal component is negligible.

It has been stressed by Landau that the properties of the two regions, below the λ-point, can be understood if one looks at the dispersion relation satisfied by the excitations in He II. The excitation that is relevant at very low temperatures is the phonon, whose dispersion relation is

$$\omega^{\mathrm{ph}}(p) = v_s p \tag{2.1}$$

where v_s is the sound speed and p is the phonon momentum. Since superfluidity can be understood from the dispersion relation (2.1), superfluidity is closely related to the existence of phonons.

The other type of quantum excitation, relevant in understanding the behavior of He^4 in the two fluid region, was named a roton by Landau,[14] its discoverer. The roton dispersion relation was postulated by Landau to be of the form

$$\omega^{\mathrm{r}}(p) = \frac{(p - p_0)^2}{2\mu} + \Delta \tag{2.2}$$

where Δ (the energy gap), μ (the effective mass) and p_0 are phenomenological constants that could be obtained, in principle, from a microscopic theory.

The spectrum of excitations should be inferred from the solutions of the quantum mechanical many-body problem. For an aggregate of roughly 10^{22} particles this is an almost impossible task. In spite of this Feynman showed[15] that one can throw considerable light on the problem by using physical intuition to obtain a qualitative picture of the wavefunctions. In Feynman's approach the dispersion relation of the excitations can be inferred from the form factor for scattering of neutrons from the liquid and thus can be inferred experimentally.

Although the spectrum of excitations is by far the most relevant problem to be settled for superfluid Helium-4, there are yet other features that any microscopic theory should account for.

2.2. *Ground state energy*

The ground state of liquid Helium is a N-body bound state that has a self-determined equilibrium density in the absence of external pressure. One should be able to write, for the energy of the ground state,

$$E = E(\rho) \tag{2.3}$$

and find the equilibrium density from the condition

$$\left.\frac{dE}{d\rho}\right|_{\rho=\rho_0} = 0\,. \tag{2.4}$$

For liquid He, $E(\rho_0)$ is the binding energy of the particles and is equivalent to the latent heat of vaporization at $T = 0°$ K.

2.3. *Binding energy*[2]

Since Helium-4 is a N-body bound state, liquid Helium has a binding energy per atom at absolute zero. This binding energy should be determined from the condition

$$\varepsilon = \frac{E(\rho_0)}{N} \tag{2.5}$$

with ρ_0 given by (2.4). The binding energy is an example of Helium phenomena for which the two-fluid model alone is simply incapable of providing a suitable description. London discussed various approaches to this problem in his book.[2] The values for the Helium binding energy is[2]

$$\varepsilon = -13\frac{\text{cal}}{\text{mol}} = -5.5\frac{10^{-4}\,\text{eV}}{\text{Atom}}\,. \tag{2.6}$$

2.4. *Helium density*

In some approaches[4,5] to the imperfect Bose gas system at very low temperatures the Helium density, given by

$$\rho_{\text{He}} = 16.10^8\frac{\text{Atom}}{(\text{eV})^{-3}} = 2.97\ 10^{22}\frac{\text{Atom}}{(\text{Cm})^3} \tag{2.7}$$

or the interatomic distance d

$$d^{-1} = (\rho_{\text{He}})^{1/3} \tag{2.8}$$

is taken to be a fundamental parameter. As explained below, in order to understand, from a microscopic theory, why Helium is liquid at very low temperatures it is also essential to get from this theory the Helium density. In principle this density should also be inferred from (2.4).

2.5. *Quantum liquid*

Why is He^4 still liquid at $T = 0$ at zero external pressure? Tisza's approach[16] to this problem imposes two postulates, one of which is analogous to (2.4). Since the He–He interaction potential is shallow and has short range, then, as a result of the small Helium mass, zero-point fluctuations prevent Helium-4 from being a solid (the lattice "melts"). If on the other hand we get from a microscopic theory the Helium density given by (2.4), then from (2.8) we should conclude that it cannot be a gas (the interparticle distance is typical of a solid). In this way, by looking at the features of the potential and from (2.8) we should understand why Helium is liquid at zero temperature.

2.6. *Critical velocity*

Superflow in Helium liquid is subject to a restriction which limits the velocity of the superfluid to values smaller than a "critical velocity". An upper bound to this critical velocity can be obtained from the knowledge of the excitation spectrum. This upper bound is[11]

$$v_c < \left(\frac{\omega(\mathrm{p})}{p}\right)_{\min} \tag{2.9}$$

where $(\omega(\mathrm{p})/p)_{\min}$ stands for the condition

$$\frac{d}{dp}\left(\frac{\omega(\mathrm{p})}{p}\right)_{p=p_{\min}} = 0 \,. \tag{2.10}$$

The determination of the spectrum is thus important in order to get expressions for the superfluid flow velocity limit.

2.7. Two-fluid description

As pointed out earlier, below the λ-point Helium-4 behaves as a mixture of two fluids. Below 1.1° K the so-called normal component is negligible. There are many phenomenologically successful applications of the two-fluid model. However the understanding of the two-fluid nature as well as a well-defined scheme for computing the thermodynamical properties of normal and/or superfluid components has not been achieved yet. In this context it is important to recall that the proposers of the two-fluid model were not in agreement with regard to the nature of the two-fluid model.[17] While in Landau's view the normal component is associated to the excitations, in Tisza's view the normal component was composed of Helium atoms.

2.8. Critical temperature

Any attempt to describe Helium-4 should predict a transition at $T_\lambda = 2.18°$ K. The determination of the critical temperature is a longstanding problem in superfluid

Helium-4. In the pioneering work of London,[1] in which he suggested that the λ-transition can be attributed to the macroscopic occupation of the $\mathbf{p} = 0$ state (Bose–Einstein condensation), he made a successful prediction for T_λ. London's result was

$$T_c^{\mathrm{D}} = 3.13^\circ \mathrm{K} . \tag{2.11}$$

2.9. *Abnormal transport coefficients*

It is known that the heat conductivity of superfluid Helium just below the λ-point is 5 orders of magnitude larger than the normal liquid (He^{I}). This large heat conductivity prevents the establishment of temperature differences in superfluid Helium and this is the basis for understanding some surprising properties of Helium-4.[10] The viscosity coefficient of the superfluid component is negligible. the viscosity coefficient of the fluid as a whole suffers a drop of about 1/3 of the normal liquid just below the λ-point.

2.10. *Thermomechanical and mechanocaloric effects*

Thermomechanical and mechanocaloric effects are among the most remarkable properties of superfluid Helium. The fact that a temperature difference (ΔT) gives rise to a pressure difference (ΔP) is the basic feature of the thermomechanical effect. The relation between ΔP and ΔT has been deduced by London in 1939[1] on the basis of essentially reversible thermodynamics. His result is known as London's relation

$$dP = \frac{S}{V} dT . \tag{2.12}$$

In this paper we shall deal with a microscopic model that allows us to make specific predictions for the physical quantities discussed here.

3. B. E. Condensation and London's Relation

3.1. *Field theoretical approach*[7,18,19]

In this section we shall review the field theoretical approach to the description of the Bose–Einstein condensation. In this case we shall be concerned with obtaining the thermodynamics of the system in terms of the Green's Functions at zero momentum. The most relevant result in this section is the derivation of London's relation from Bose–Einstein condensation.

Within the field theoretical approach one writes the partition function as a sum over field theoretical configurations satisfying periodic boundary conditions[19]

$$\psi(\mathbf{x}, \tau) = \psi(\mathbf{x}, \tau + \beta) \tag{3.1}$$

as

$$Z = \iint \mathcal{D}\psi\mathcal{D}\psi^* e^{-\int_0^\beta d\tau \int d^3x \mathcal{L}[\psi, \partial_\mu \psi]} . \tag{3.2}$$

where β in (3.1) and (3.2) is the inverse of the temperature $\beta = 1/T$. The Lagrangian density $\mathcal{L}$ can be written as a kinetic term (K) plus an interaction term V as

$$\mathcal{L} = \psi^* K \psi - V(\psi\psi^*) . \tag{3.3}$$

For a nonrelativistic theory one writes

$$K = \frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m} \tag{3.4}$$

whereas in the relativistic case one writes

$$K = \frac{\partial^2}{\partial\tau^2} - \nabla^2 . \tag{3.5}$$

From (3.2) one gets the free energy

$$F(T, V) = -\beta^{-1} \ln Z . \tag{3.6}$$

The whole thermodynamics can be obtained from the thermodynamical potential above.

3.2. *Bose–Einstein condensation*

The relevance of Bose–Einstein condensation for understanding the properties of liquid He^4 was pointed out first by London[1] who, besides proving Bose–Einstein condensation for an ideal gas, suggested that the peculiar phase transition that liquid Helium undergoes at 2.18° be regarded as a Bose–Einstein condensation phenomenon. The temperature for which a finite fraction of all Helium atoms will be assembled in the lowest energy state, in the ideal gas approximation, is very close to the critical temperature of the λ-point. The hypothesis that He^4, in the superfluid phase, has to do with Bose–Einstein condensation received a boost after the seminal work of Bogoliubov[8] who showed that the "phonon" part of spectrum of excitations of He^4 follow directly from Bose–Einstein condensation of an imperfect gas.

Within the field theoretical context Bose–Einstein condensation is intimately related to the spontaneous breakdown $U(1)$ symmetry since in the broken symmetry phase the expectation of value of the scalar field ψ is different for zero[7,20]

$$\langle\psi\rangle = \sqrt{\rho_c} . \tag{3.7}$$

Under condition (3.7) we say, on the other hand, that the fluid is a superfluid. Actually in Anderson's definition[21] a superfluid is a fluid for which $U(1)$ symmetry is broken spontaneously and this is the content of (3.7). Other approaches to superfluidity are based essentially on properties of Green's function analogous to (3.7).[22,23] The description of Bose–Einstein condensation is then crucial for the understanding of the superfluid phase of Helium-4.

In order to study Bose–Einstein condensation we separate out the uniform (in space) field theoretical configuration. In this way one writes, in analogy with Bogoliubov's approach,[7,8]

$$\psi = \psi_0 + \psi' \ . \tag{3.8}$$

By substituting (3.8) into (3.2) one can write the free energy as a function of the Bogoliubov condensate,[7] that is,

$$F = F(T,\ V,\ \psi_0) = F(V,\ T,\ \rho_0) \ . \tag{3.9}$$

Since the number of particles with zero-momentum is given by

$$N_0 = V\rho_0 = V\psi_0^*\psi_0 \ , \tag{3.10}$$

the dependence of F on ψ_0 is equivalent to the dependence of F on N_0. As suggested by Glassgold, Kaufman and Watson[24] the important contribution comes from the term that minimizes F, that is,

$$\left.\frac{\partial F}{\partial \rho_0}\right|_{\rho_c} = 0 \ . \tag{3.11}$$

Equation (3.11) is the basic equation for the determination of the occupation of the $\mathbf{p} = 0$ state.

The system described here is expected to exhibit two phases. In the noncondensed phase

$$\rho_c = 0 \tag{3.12}$$

whereas in the condensed phase

$$\rho_c = \rho_c\left[\frac{N}{V},\ T\right] = \rho_c[\rho_{\mathrm{He}},\ T] \ . \tag{3.13}$$

The critical temperature is the one that distinguishes the two phases. ρ_c plays the role of the order parameter. One defines the critical temperature as the one for which

$$\rho_c\left[\frac{N}{V},\ T_c\right] = 0 \ . \tag{3.14}$$

Within field theory one can get all the relevant information on Bose–Einstein condensation phenomena from the field theoretical Green's functions in momentum space computed at zero-momenta.[18] By following Ref. 7 one can write, in field theory,

$$F(\rho_0, V, T) = V\Gamma(\rho_0, T) \tag{3.15}$$

where $\Gamma(\rho_0, T)$ is the generating functional of the one-particle irreducible Green's functions at zero momenta, that is,

$$\Gamma(\rho_0, T) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{1}{m!}\frac{1}{n!}\tilde{\Gamma}^{(n,m)}(0\ldots0,\,0\ldots0,\,T)\rho_0^{\frac{m+n}{2}} \tag{3.16}$$

where $\tilde{\Gamma}^{(n,m)}$ stands for the Fourier transform of the one-particle irreducible Green's functions at zero momenta,

$$\tilde{\Gamma}^{(n,m)}(0,\,\ldots\,,0,\,T) = \tilde{\Gamma}^{(n,m)}(p_1\ldots p_m)\big|_{p_1=p_2=\cdots=p_n=0}\,. \tag{3.17}$$

As far as condensation phenomena is concerned $\Gamma(\rho_0, T)$ is the relevant potential. The whole thermodynamics can be inferred from $\Gamma(\rho_0, T)$.

From $\Gamma(\rho_0, T)$ one gets the occupation of the $\mathbf{p} = 0$ (ρ_c) from the equation

$$\left.\frac{\partial(\Gamma(\rho_0, T))}{\partial\rho_0}\right|_{\rho_0=\rho_c} = 0 \tag{3.18}$$

whereas the pressure (P), entropy (S), internal energy (U) and specific heat (C_V) are obtained from the equations

$$P = -\Gamma[\rho_c, T] \tag{3.19}$$

$$S = V\left[-\frac{\partial\Gamma(\rho_0, T)}{\partial T}\right]_{\rho_0=\rho_c} \tag{3.20}$$

$$U = V\left[\Gamma[\rho_0, T] - T\frac{\partial\Gamma(\rho_0, T)}{\partial T}\right]_{\rho_0=\rho_c} \tag{3.21}$$

$$C_V = \frac{dU}{dT}\,. \tag{3.22}$$

Equations (3.18) and (3.19)–(3.22) are the basic equations of condensation phenomena.

3.3. *Equation of state and London's relation*

Although Eqs. (3.19)–(3.22) depend on the thermodynamical potential Γ which can be computed explicitly only under certain approximations, (as we shall see in the next section), one can derive two relevant properties without the explicit knowledge of Γ. These properties do not depend on the dynamics. They follow from the implicit assumption that there is only the superfluid component present in the system.

The first property that one expects in Helium superfluid, if such a phenomena is related to Bose–Einstein condensation, is that the state of superfluid Helium should be characterized by zero Gibbs energy. In fact, it follows from (3.19)–(3.22) that

$$U + PV = TS(T) \, . \tag{3.23}$$

From (3.23) it follows that the Gibbs energy is zero in the condensed phase:

$$G = 0 \, . \tag{3.24}$$

As a result of the third law of thermodynamics one expects that

$$\lim_{T\to 0} TS(T) = 0 \, . \tag{3.25}$$

By dividing (3.23) by N and then taking the zero temperature limit, we can predict the equation of state

$$\frac{P}{\rho_{\mathrm{He}}} = -\varepsilon \tag{3.26}$$

where ε is the binding energy per atom.

The static fountain effect can be explained from London's relation

$$dP = \frac{S}{V} dT \, . \tag{3.27}$$

We can see that this relation follows as a consequence of our basic equations describing the superfluid phase. In fact, if one derives the pressure as a function of T, one gets (independently of the details of the dynamics), from (3.19) that

$$\frac{dP}{dT} = \frac{\partial\Gamma(\rho_c T)}{\partial\rho_c}\frac{d\rho_c}{dT} - \left(\frac{\partial\Gamma(\rho_0, T)}{\partial T}\right)_{\rho_0=\rho_c} \, . \tag{3.28}$$

From Eqs. (3.19), (3.20) and (3.21), London's relation (2.12) follows. We have shown that London's relation follows from equilibrium thermodynamics when Bose–Einstein condensation occurs. London's relation follows also from (3.23) since

$$dG = V\,dP - S\,dT \, . \tag{3.29}$$

4. Effective Hamiltonian and Loop Expansion

In this section we shall build an effective Hamiltonian that allows us to make predictions by using the field theoretical approach. This effective Hamiltonian does not contain any free parameter, being a microscopic model. The method on which we will rely, in order to make predictions, is the Loop Expansion approach. The justification for using the method to superfluidity as well as a brief discussion of the method will also be presented here.

4.1. *Effective Hamiltonian*

In order to get the thermodynamics of an imperfect gas, we shall replace the Hamiltonian of the system by an effective Hamiltonian. Within the field theoretical method this amounts to making the replacement.

$$H_{\text{int}}[\psi] \to H_{\text{eff}}[\psi] \tag{4.1}$$

where ψ is the complex boson field and $H_{\text{int}}[\psi]$ can be written, for particles that interact through binary forces whose potential is $V(\mathbf{x}-\mathbf{x}')$, as[18]

$$\begin{aligned} H_{\text{int}}[\psi] &= \frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{x}'\psi^*(\mathbf{x},\,\tau)\psi(\mathbf{x},\,\tau)V(\mathbf{x}-\mathbf{x}')\psi^*(\mathbf{x}',\,\tau)\psi(\mathbf{x}',\,\tau) \\ &= \frac{1}{2}\int d^3\mathbf{x}\int d^3\mathbf{x}\rho(\mathbf{x},\,\tau)V(\mathbf{x}-\mathbf{x}')\rho(\mathbf{x}',\,\tau) \end{aligned} \tag{4.2}$$

where

$$\rho \equiv \psi^*(\mathbf{x},\,\tau)\psi(\mathbf{x},\,\tau)\ . \tag{4.3}$$

The justification for the replacement (4.1) is that the scattering of particles at low momenta is insensitive to the shape of the potential.[4,5] Within the shape independent approximation we can replace the potential by any other having the same scattering length.

In order to find the effective Hamiltonian suited to He^4 at very low temperatures we show in Fig. 1 the two-particle interaction potential between two Helium atoms. The essential feature of the two-body potential is that, as far as the repulsive part of the potential is concerned, the potential is typical of a hard sphere whose diameter is of the order[5,25] 2.6 Å. We then take

$$a \simeq 2.6\,\text{Å} = (0.77\cdot 10^3\,\text{eV})^{-1}\ . \tag{4.4}$$

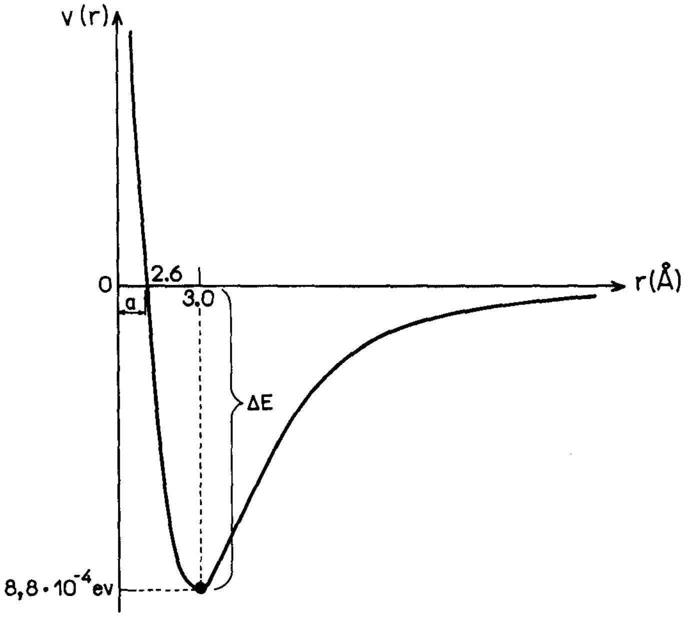

Fig. 1. Potential energy between two He-4 atoms as function of distance. The relevant parameters, ΔE and a, are shown explicitly here.

One is tempted to make the replacement (4.1) with H_{eff} given by

$$H_{\text{eff}}[\psi] = \frac{\lambda}{2} \int d^3\mathrm{x}(\psi^*(\mathrm{x}, t)\psi(\mathrm{x}, t))^2 \,. \tag{4.5}$$

The parameter λ is then fixed by condition (1.1), yielding

$$\lambda = \frac{4\pi a}{m} \,. \tag{4.6}$$

The hard-sphere boson gas described by (4.5) has been extensively investigated in a series of papers by Lee, Yang and Huang.[4,26,27] If the scattering length is fixed by (4.6), then, apparently, there is no more room for adding terms to the effective Hamiltonian. However the scattering length fixed criterion together with the requirement of particle number conservation allows for yet another term. The most general effective Hamiltonian preserving $U(1)$ symmetry and the scattering length as given by (4.6) is

$$\mathcal{H}_{\text{eff}} = -\Delta E \psi^* \psi + \frac{2\pi a}{m}(\psi^* \psi)^2 \,. \tag{4.7}$$

where ΔE is (4.7) is a term with energy dimension. The basic content of (4.7) is that the repulsive core of the two-body potential between Helium-4 atoms is replaced by the hard sphere potential (4.5) whereas the attractive interparticle potential is replaced by a single particle term. One then writes

$$\begin{aligned}\mathcal{H}_{\text{eff}} &= \mathcal{H}^1[\psi\psi^*] + \frac{\lambda}{2}(\psi^*\psi)^2\\ &= -\Delta E\psi^*\psi + \frac{\lambda}{2}(\psi^*\psi)^2\end{aligned} \tag{4.8}$$

where $\mathcal{H}^1$ is the single particle contribution.

The single particle term $(-\Delta E\psi^*\psi)$ in (4.7) represent a constant uniform negative potential that acts on the individual particles. The justification for adding such a term can be traced back to the seminal papers of Huang, Lee and Yang.[4,26] In fact, in the third paragraph of Ref. 26, Lee, Huang and Yang argue that "[t]he interaction between real He atoms contains besides a hard repulsive core, also an attractive interaction outside of the core. This attractive interaction is responsible for many properties of the He liquid. For example, the ground state of a system of He atoms is known to have a negative energy corresponding to a binding energy per He atom of (k × 7°), as determined from the experimental vapor pressure curve near the absolute zero of temperature. Such a bound system certainly owes its origin to the attractive force. The strength of the attractive force also determines the density of the He atoms in the ground state. Now at this density the total attractive potential that a He atom experiences from its neighbors is not expected to fluctuate very much. This fact suggests the following approximate picture: One replaces the attractive interparticle forces by a constant uniform negative external potential that acts on the individual particles, the repulsive core is retained".

As suggested by Lee, Huang and Yang[26] the single particle term takes into account the attractive nature of the potential. In order to identify the parameter ΔE in (2.7) we just recall that a single particle in the fluid is under the action of a potential given by

$$H^{(1)}[\psi\psi^*] = \int V(\mathbf{x}-\mathbf{x}')\psi^*(\mathbf{x}')\psi(\mathbf{x}')d^3\mathbf{x}' \ . \tag{4.9}$$

In order to further simplify H^1 in (4.9) we just recall that in the low temperature limit a single particle is insensitive to some details of the potential. In fact, within the semiclassical approximation the only relevant region of the potential is the point in which the potential has a minimum. If one expands V around the minimum one gets

$$H^{(1)} \equiv \int \mathcal{H}^1[\psi^*\psi]d^3\mathbf{x}' = \int V_{\text{min}}(\psi^*\psi)(x')d^3\mathbf{x}' \ . \tag{4.10}$$

From (4.8) and (4.10) it follows that

$$\Delta E = V_{\min} \ . \tag{4.11}$$

Expression (4.11) allows us to identify ΔE with the depth of the interatomic He^4 potential. From Fig. 1

$$\Delta E = 8.81 \cdot 10^{-4}\,\text{eV} = 10.2^\circ\,\text{K} \tag{4.12}$$

follows.

Model (4.7) is a prototype of theories that exhibit a spontaneous breakdown of symmetry. The symmetry that is spontaneously broken in model (4.7) is the $U(1)$ symmetry that, as is well known,[20] is the symmetry that is broken in the superfluid phase of He-4. One can then predict two phases for He^4. The one for which the $U(1)$ symmetry is broken corresponds to superfluid He^4 at very low temperatures.[28]

Model (4.7) is just an extended version of the pioneering model proposed and studied by Lee, Yang and Huang.[4,26] In this context the model proposed here is an attempt to take into account, through the single particle term in (4.7), the attractive part of the two-body potential.

Since the description of superfluid Helium will be based on model (4.7), we shall end this section by analyzing the validity of applying the effective Hamiltonian (4.7) to liquid Helium. It is well known[4,27] that the purely repulsive hard sphere model (4.5) is valid in the low density limit. The low density condition is

$$\rho_{\text{He}} a^3 \equiv \left(\frac{a}{d}\right)^3 \ll 1 \tag{4.13}$$

where ρ_{He} is the Helium density and a is the effective diameter of the potential. One can take (4.13) as the condition for the validity of our model (4.7) since we do not expect that the single term added to the Lee–Yang hard sphere model will change the basic limitation (4.13).

The values of a and ρ_{He} used throughout this paper are those given in expressions (1.2) and (2.7). By inserting these values in (4.13) we get

$$\rho_{\text{He}} a^3 = (0.71)^3 = 0.35 \ . \tag{4.14}$$

Although $\rho_{\text{He}} a^3$ is smaller than 1, it does not satisfy (4.13), strictly speaking. For this reason the hard sphere gas model is sometimes viewed as only qualitatively relevant to the description of liquid Helium. However, the fact that $\rho_{\text{He}} a^3$ is small makes us hopeful that the hard sphere gas model applies to the description of superfluid Helium. This description will be presented in the following sections.

4.2. *Loop expansion*

In his pioneering work on Bose–Einstein condensation in an imperfect gas Bogoliubov[8] argued that when B. E. condensation occurs one can treat the creation and destruction operators $b(0)$ and $b^{+}(0)$ appearing in the Fourier expansion of the field operator

$$\psi = V^{-1/2}\sum b(\mathbf{k})e^{i\mathbf{kx}} = \frac{b_0}{\sqrt{V}} + \psi' \equiv \psi_c + \psi \tag{4.15}$$

as classical variables. This means that one can replace these variables in zero-mode amplitudes by c numbers, i.e., $b_0 = b_0^{+} = N_0^{1/2}$, N_0 being the number of particles in the condensate. The validity of treating these variables as classical (c-numbers) is more reliable as the condensate becomes more and more occupied.

The fact that ψ_0 in (3.1) can be considered as a classical variable allowed him (and later on generalized by Belyaev[22]) to propose an approximation scheme for calculations when B. E. condensation occurs. His scheme is the semiclassical method. Within the semiclassical method one expands all physical quantities (as will be done for the free energy in the following) in powers of $\hbar$.

In this way, if one expands (2.30) in powers of $\hbar$ one can write

$$\Gamma(\rho_0, T) = \sum_{K=0}^{\infty} \Gamma^{(K)}(\rho_0, T) \tag{4.16}$$

where $\Gamma^{(K)}$ is the Kth term in a series expansion of Γ in powers of $\hbar$. It can be shown that, in field theory, the contribution to $\Gamma^{(K)}$ comes from graphs having K-loops; that is, for computing $\Gamma^{(k)}$ in (4.16) one considers only those irreducible Green's function having K-loops. For this reason the semiclassical expansion is also referred to as loop expansion.

The loop expansion provides a systematic as well as proper method for dealing with Bose–Einstein condensation. As pointed out in Ref. 7 the method is richer and more appropriate than the perturbative approach or the low density approximation.

At the classical level (zero-loop approximation) one can write,[7] for model (4.7),

$$\Gamma^{0}(\rho_0, T) = -\Delta E \rho_0 + \frac{\lambda}{2}\rho_0^2 \,. \tag{4.17}$$

One can compute Γ explicitly up to one loop level. The answer, for our model, can be inferred from Toyoda's papers.[29] However, in order to understand the roton spectrum, we will present an alternative derivation of Toyoda's result later.

4.3. *Semiclassical approximation*

The partition function for our model is, from (3.2), (3.3) and (4.7),

$$Z(V,T) = \iint \mathcal{D}\psi\mathcal{D}\psi^*$$
$$= \exp\left[-\int_0^\beta d\tau \int d^3x \left\{\psi^*\left(-\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi + \Delta E\psi^*\psi - \frac{\lambda}{2}(\psi^*\psi)^2\right\}\right] . \tag{4.18}$$

The semiclassical method is based upon the hypothesis that under some physical conditions the classical configuration defined as the minimum of the action,

$$\left(-\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right)\psi_c + \Delta E\psi_c - \lambda(\psi_c^*\psi_c)\psi_c = 0 \tag{4.19}$$

gives the dominant contribution.

For uniform configurations Eq. (4.19) implies

$$\psi_c^*\psi_c = \frac{\Delta E}{\lambda} . \tag{4.20}$$

If we expand ψ in (4.18) around the classical configuration (4.19)

$$\psi = \psi_c + \eta . \tag{4.21}$$

Then, if we expand the exponents in (4.18) in powers of η one gets, formally,

$$Z = e^{-\beta F_c} \cdot \int \mathcal{D}\eta\mathcal{D}\eta^* \exp\left\{-\int d\tau d^3x \frac{1}{2}(\eta^*, \eta)G'\begin{pmatrix}\eta\\ \eta^*\end{pmatrix}\right\} \tag{4.22}$$

where G' is a two-by-two matrix given by

$$G' = \begin{bmatrix} K - \Delta E + 2\lambda\psi_c^*\psi_c & \lambda\psi_c^{2*} \\ \lambda\psi_c^2 & K - \Delta E + 2\lambda\psi_c^*\psi_c \end{bmatrix} \tag{4.23}$$

where

$$K = -\frac{\nabla^2}{2m} + \frac{\partial}{\partial\tau} \tag{4.24}$$

and

$$F_c = \frac{1}{\beta}\int_0^\beta \int d^3x \left[\psi_c^*\left[\frac{\partial}{\partial\tau} - \frac{\nabla^2}{2m}\right]\psi_c + \Delta E\psi_c^*\psi_c - \frac{\lambda}{2}(\psi_c^*\psi_c)^2\right] . \tag{4.25}$$

For constant ψ_c, those satisfying (4.20), we get

$$F_c = \frac{\Delta E^2}{2\lambda} \,. \tag{4.26}$$

Formally the free energy can be written, in the semiclassical approximation, as

$$F = F_c + \beta^{-1} \ln \det G' \tag{4.27}$$

where F_c is defined in (4.45) and G' is defined in (4.23).

5. One-Loop Approximation

The specific predictions that will be made in the next three sections, for superfluid Helium-4, rely on the one-loop approximation. The treatment of Bose–Einstein condensation within the one-loop approximation will be discussed in this section. Up to the one loop approximation one can write

$$\begin{aligned} F &= F_0 + F_1 \\ &= V[\Gamma^{(0)} + \Gamma^{(1)}] \end{aligned} \tag{5.1}$$

where F_0 is the zero loop approximation that, by using (4.17), is given by

$$F_0(\rho_0, V, T) \equiv V\left[-(\Delta E)\rho_0 + \frac{\lambda}{2}\rho_0^2\right] \,. \tag{5.2}$$

$\Gamma^{(1)}$ in (5.1), represents the first quantum corrections. After substituting (3.8) into (3.2) with the interaction given by (4.7) and then expanding in ψ', taking into account only quadratic terms, we end up with a quadratic functional integral in ψ'. The result is

$$\begin{aligned} \Gamma^{(1)}(\rho_0, T) &\equiv \int \mathcal{D}\psi' \mathcal{D}\psi' \exp -\frac{1}{2}\int (\psi'^*, \psi') G \begin{pmatrix} \psi' \\ \psi'^* \end{pmatrix} \\ &\equiv (\det G)^{-1} \end{aligned} \tag{5.3}$$

where G is a matrix element which in momentum space has matrix elements given by

$$G(n, \mathbf{k}, T, \rho_0) = \begin{bmatrix} \tilde{K}(n, \mathbf{k}, \beta) - \Delta E + 2\lambda\rho_0 & \lambda(\psi_0^*)^2 \\ \lambda(\psi_0^2) & \tilde{K}^*(n, \mathbf{k}, \beta) - \Delta E + 2\lambda\rho_0 \end{bmatrix} \tag{5.4}$$

where $\tilde{K}$ is the Fourier transform of the kinetic term K. In the nonrelativistic case $\tilde{K}$ is given by

$$\tilde{K} = \frac{2in\pi}{\beta} + \frac{\mathbf{k}^2}{2m} \equiv i\omega_n + \frac{\mathbf{k}^2}{2m} \tag{5.5}$$

whereas in the relativistic case, one has

$$\tilde{K}(n, \mathbf{k}, \beta) = \left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2 \, . \tag{5.6}$$

One can diagonalize the matrix G in (5.4) by using a Bogoliubov's transformation and write G in a diagonal form:

$$G \rightarrow G' = \begin{pmatrix} \lambda^+(\mathbf{k}, \rho_0, \beta) & 0 \\ 0 & \lambda^-(\mathbf{k}, \rho_0, \beta) \end{pmatrix} \tag{5.7}$$

where the eigenvalues $\lambda^\pm$ are given by

$$\lambda^\pm = \operatorname{Re} \tilde{K}(n, \mathbf{k}, \beta) - \Delta E + 2\lambda\rho_0 \pm \sqrt{\lambda^2\rho_0^2 - (\operatorname{Im} \tilde{K}(k, n, \beta))^2} \, . \tag{5.8}$$

with $\lambda^\pm$ given by (5.8) one can write, up to the one loop level,

$$\frac{F}{V}(\rho_0, V, T) = -(\Delta E)\rho_0 + \frac{\lambda}{2}\rho_0^2 + \frac{1}{2\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln \lambda^+\lambda^- \, . \tag{5.9}$$

This expression is the basic one in order to study superfluid Helium at the one loop level.

In the nonrelativistic case

$$\lambda^+\lambda^- = \left(\frac{2n\pi}{\beta}\right)^2 + \left[\left(\frac{\mathbf{k}^2}{2m}\right) - \Delta E + 2\lambda\rho_0\right]^2 - \lambda^2\rho_0^2 \tag{5.10}$$

whereas in the relativistic case

$$\lambda^+\lambda^- = \left[\mathbf{k}^2 - \Delta E + 2\lambda\rho_0 + \left(\frac{2n\pi}{\beta}\right)^2\right]^2 - \lambda^2\rho_0^2 \, . \tag{5.11}$$

In both cases we can perform the sum over n by using the identity

$$\frac{1}{\beta}\sum_n \ln\left(\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right) = \omega + \frac{2}{\beta}\ln\left(1 - e^{-\beta\omega}\right) \, . \tag{5.12}$$

The general structure of F up to the one loop approximation is, by using (5.12),

$$F(\rho_0, V, T) = V\left[-\Delta E\rho_0 + \frac{\lambda}{2}\rho_0^2 + \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\omega(\mathbf{k}, \rho_0)}{2}\right] + \frac{V}{\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ln\left(1 - e^{-\beta\omega(\mathbf{k}, \rho_0)}\right) \tag{5.13}$$

where in (5.13) we have separated out the zero temperature contribution (the temperature-independent part in (5.13) from the temperature-dependent contribution. ω in (5.13) can be inferred from (5.10) and (5.11).

The zero temperature contribution is divergent and requires renormalization. Up to the one loop approximation one has to add the usual counterterms to the Hamiltonian (4.7). One has also to remove a zero point energy contribution in (5.13). The equation for the occupation of the $\mathbf{p} = 0$ state is

$$\Delta E = \lambda\rho_c + \frac{1}{2}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{d\omega(\mathbf{k}, \rho_0)}{d\rho_0}\bigg|_{\rho_0=\rho_c} + \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\frac{d\omega}{d\rho_0}}{e^{\beta\omega} - 1}\bigg|_{\rho_0=\rho_c} . \tag{5.14}$$

The equation (5.14) is also divergent but these divergences can be eliminated by the usual procedure of adding counterterms.[7,18] After renormalizing (5.14) we could in principle discuss the depletion of the condensate as a result of quantum effects.[30] In this paper we will work within the semiclassical approach for computing the thermodynamics of He-4. Within the semiclassical method the divergences can be removed by including a zero-point energy subtraction (zero point energy of phonons).

The semiclassical method presented here, using the functional method, is equivalent to the one originally proposed by Bogoliubov[8] using the operator approach.[7] This equivalence is better seen in the Bogoliubov transformation leading from (5.4) to (5.7).

5.1. *Relativistic case*

In the relativistic case one can consider two cases. In the first case

$$-\Delta E < 0 \tag{5.15}$$

and the theory exhibits spontaneous breakdown of $U(1)$ symmetry. The condition

$$\frac{dF}{d\rho_0}\bigg|_{\rho_c} = 0 \tag{5.16}$$

determines the occupation of the $p = 0$ states as a function of temperature. At the classical level one has

$$\rho_c^{(0)} = \frac{\Delta E}{\lambda} \tag{5.17}$$

Since in the relativistic case

$$\begin{aligned} \lambda^+(\mathbf{k}, n, \rho_0) &= \left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{k}^2 - \Delta E + 3\lambda\rho_0 \\ \lambda^-(\mathbf{k}, n, \rho_0) &= \left(\frac{2\pi n}{\beta}\right)^2 + \mathbf{k}^2 - \Delta E + \lambda\rho_0 \ , \end{aligned} \tag{5.18}$$

the spectrum of excitation involves two types of particles having masses M_1 and M_2, and these masses are given by the square root of $\lambda^{\pm}(n = 0, \mathbf{p} = 0)$, one gets

$$\begin{aligned} M_1^{\rm BS} &= \lambda^+(n = 0, \mathbf{k} = 0, \rho_c) = \sqrt{2\Delta E} \\ M_2^{\rm BS} &= \lambda^-(n = 0, \mathbf{k} = 0, \rho_c) = 0 \ ; \end{aligned} \tag{5.19}$$

that is, one gets a massive particle and one massless Goldstone boson.[31]

The thermodynamics of the system at very low temperatures will be entirely controlled by the Goldstone boson mode. One gets explicitly in this phase

$$\begin{aligned} \frac{1}{V}F(\rho_0, T) = -\Delta E\rho_0 + \frac{\lambda\rho_0{}^2}{2} + \frac{1}{2\beta}\sum\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\ln\left(\left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2 + 2\Delta E\right)\right. \\ \left. + \ln\left(\left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2\right)\right] . \end{aligned} \tag{5.20}$$

In the symmetry restored phase, characterized by

$$-\Delta E > 0 \tag{5.21}$$

the spectrum is also well-known. These are two particles with the same mass $\sqrt{\Delta E}$

$$M_1 = M_2 = \sqrt{\Delta E} \ . \tag{5.22}$$

These particles obey the dispersion relation

$$\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + \Delta E} \ . \tag{5.23}$$

The spectrum for the symmetric phase ($\Delta E > 0$) can be obtained directly from the broken symmetry phase if one makes use of a simple approach to symmetry restoration. This will be explained in the next section.

5.2. *Simple approach to symmetry restoration*

We shall start now from the broken symmetry phase in which the matrix G is diagonalized, and write

$$G' = \begin{bmatrix} \left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2 + (\sqrt{2\Delta E})^2 & 0 \\ 0 & \left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2 \end{bmatrix} . \tag{5.24}$$

Let us now introduce a coupling between the massive particles of mass $\sqrt{2\Delta E}$ and the Goldstone mode of the form

$$G' \to G'' = \begin{bmatrix} \left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2 + 2\Delta E & i\Delta E \\ i\Delta E & \left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k} \end{bmatrix} . \tag{5.25}$$

The new eigenvalues are

$$\lambda^+(n, \mathbf{k}) = \lambda^-(n, \mathbf{k}) = \left(\frac{2n\pi}{\beta}\right)^2 + \mathbf{k}^2 + \Delta E \ . \tag{5.26}$$

This is result (5.23).

The spectrum of the symmetric phase is thus obtained as a result of the coupling (5.25) between the two modes (5.19).

5.3. *Validity of the semiclassical approach*

As pointed out earlier, the semiclassical approach is supposed to be a good method when one is dealing with Bose–Einstein condensation. The method is more reliable as the condensate becomes more and more occupied. This is one lesson that can be learned from Bogoliubov's work.[8]

Within the semiclassical method an arbitrary thermodynamic variable G will be written (as we did up to the one loop approximation in the section) as

$$G = \sum_{K=0}^{\infty} G^{(K)} \tag{5.27}$$

where $G^{(K)}$ is the Kth order of the semiclassical expansion. One expects that the validity of computing up to, say, the jth order will be justified if

$$\frac{G^{(j+1)}}{G^{(j)}} < 1 \ . \tag{5.28}$$

There is no systematic way of checking the validity of the method in field theory. However, up to the one loop level one can always compare the one loop prediction with the zero-loop result. The one-loop results come from the determinant (5.3), whereas the zero-loop results come from the "classical" term (as in (5.2)). It is then straightforward to compare the quantum corrections with the classical result. Let us take the pressure as an example. The one loop result ($P^{(1)}$) divided by the zero-loop result ($P^{(0)}$) is, from (6.19),

$$\frac{P^{(1)}}{P^{(0)}} = \frac{8\pi^2 10^{13} T^4}{9 \cdot (5.7)} \,. \tag{5.29}$$

One can easily check that for $T < 3^\circ$ K ($T < 25\ 10^{-5}\,\mathrm{eV}$) we get

$$P^{(1)} \ll P^{(0)} \,. \tag{5.30}$$

In this paper we will see that the semiclassical method gives a fairly good account of the superfluidity phenomenon. We have not made a systematic analysis of the role of higher order terms. This should be done in the future.

6. Superfluid Helium at very Low Temperature

Let us apply the semiclassical method to the description of superfluid Helium at very low temperatures. Since the system is nonrelativistic one can write the free energy (5.9) as

$$F(\rho_0, V, T) = V\left[-\Delta E \rho_0 + \frac{\lambda \rho_0^2}{2}\right] + \frac{1}{2\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left[\left(\frac{2n\pi}{\beta}\right)^2 + \omega^2[\mathbf{k},\, \rho_0]\right] \tag{6.1}$$

where

$$\omega^2[\mathbf{k},\, \rho_0] = \left(\frac{\mathbf{k}^2}{2m} - \Delta E + 2\lambda\rho_0\right)^2 - \lambda^2 \rho_0^2 \,. \tag{6.2}$$

After subtracting the zero point energy contribution one gets

$$F(\rho_0, V, T) = V\left[-\Delta E \rho_0 + \frac{\lambda \rho_0^2}{2}\right] + \frac{V}{\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left(1 - e^{-\beta\omega(\mathbf{k},\, \rho_0)}\right) \,. \tag{6.3}$$

The occupation of the $\mathbf{p} = 0$ (ρ_c) should be inferred from the minimization of the free energy (6.3). One gets

$$\lambda \rho_c = \Delta E - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left\{\frac{1}{e^{+\beta\omega(\mathbf{k},\, \rho_0)} - 1}\right\} \frac{d\omega}{d\rho_0}\bigg|_{\rho_0 = \rho_c} \,. \tag{6.4}$$

Equation (6.4) allows us to get the occupation of the $\mathbf{p} = 0$ state as a function of the temperature within the one-loop approximation. $\omega(\mathbf{k}, \rho_0)$ in (6.4) is a function of ρ_0 and is given by (6.2). Once ρ_c is obtained one can get, by using Eqs. (3.19)–(3.22), other thermodynamical variables. The pressure as a function of the condensate, up to the one loop level, is given by

$$P = +\Delta E \rho_c - \frac{\lambda \rho_c^2}{2} - \frac{1}{\beta} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left(1 - e^{-\beta\omega(\mathbf{k},\, \rho_c)}\right) . \tag{6.5}$$

The entropy, on the other hand, from (3.20) and (6.3), is

$$\frac{S}{V} = - \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left(1 - e^{-\beta\omega(\mathbf{k},\, \rho_c)}\right) + \beta \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega(\mathbf{k},\, \rho_c)}{e^{\beta\omega(\mathbf{k},\, \rho_c)} - 1} \tag{6.6}$$

whereas the internal energy will be

$$U = V\left[-\Delta E \rho_c + \frac{\lambda \rho_c^2}{2}\right] + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega(\mathbf{k},\, \rho_c)}{e^{\beta\omega(\mathbf{k},\, \rho_c)} - 1} \tag{6.7}$$

and

$$C_V = \beta^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\omega^2(\mathbf{k},\, \rho_c)}{e^{\beta\omega(\mathbf{k},\, \rho_c)} - 1} . \tag{6.8}$$

In order to get the thermodynamics at very low temperatures let us solve equation (6.4) in the zero temperature limit. At zero temperatures the solution to (6.4) will be given by the classical solution

$$\rho_c = \frac{\Delta E}{\lambda} = \frac{m \Delta E}{4\pi a} . \tag{6.9}$$

6.1. *Phonon Spectrum*

For ρ_c given by (6.9) one gets, from (6.2), the dispersion relation

$$\omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 \frac{\Delta E}{m} + \left(\frac{\mathbf{p}^2}{2m}\right)^2} . \tag{6.10}$$

For small momenta, one finds, from (6.10), the phonon spectrum

$$\omega^{\mathrm{ph}}(\mathbf{p}) = p\sqrt{\frac{\Delta E}{m}} , \tag{6.11}$$

which, as we shall see later, can be written as

$$\omega^{\mathrm{ph}}(\mathbf{p}) = v_s p . \tag{6.12}$$

where v_s is the sound speed.

6.2. *Thermodynamics at very low temperatures*

By using (6.9) as an approximate solution to (6.4) one gets the following results, valid in the low temperature limit:

$$\frac{U}{V}=\left[-\frac{\Delta E^2 m}{8\pi a}\right]+\frac{\pi^2}{30}\left(\sqrt{\frac{m}{\Delta E}}\right)^3 T^4 \tag{6.13}$$

$$s\equiv\frac{S}{V}=\frac{2\pi^2}{45}\left(\sqrt{\frac{m}{\Delta E}}T\right)^3 \tag{6.14}$$

$$P=\frac{m\Delta E^2}{8\pi a}+\frac{\pi^2}{90}\left(\sqrt{\frac{m}{\Delta E}}\right)^3 T^4 \tag{6.15}$$

$$\frac{C_V}{V}=\frac{4\pi^2}{30}\left(\sqrt{\frac{m}{\Delta E}}\right)^3 T^3\,. \tag{6.16}$$

If we substitute the values of the constants m, ΔE, and a given by (1.2) one gets the following predictions:

$$\frac{U}{V}=-8.3\cdot 10^4(\mathrm{eV})^4+\frac{\pi^2}{30}8\cdot 10^{18}T^4 \tag{6.17}$$

$$\frac{S}{V}=\frac{2\pi^2}{45}8\cdot 10^{18}T^3\,. \tag{6.18}$$

$$P=8.3\cdot 10^4(\mathrm{eV})^4+\frac{\pi^2}{90}8\cdot 10^{18}T^4 \tag{6.19}$$

$$C_V=\frac{4\pi^2}{30}8\cdot 10^{18}T^3\,. \tag{6.20}$$

Since the temperature dependence for the specific heat and entropy are already well-known theoretically and verified experimentally, we shall turn to the internal energy and pressure. Before doing so we would like to draw attention to two identities which follow from Eqs. (6.17)–(6.19). We get

$$VP(T)=\frac{Vm\Delta E^2}{8\pi a}+\frac{1}{4}TS(T) \tag{6.21}$$

and

$$U(T)=-\frac{m\Delta E^2}{8\pi a}+\frac{3}{4}TS(T) \tag{6.22}$$

where $S(T)$ is defined in (6.14).

6.3. *Ground state energy*

For ρ_c given in (6.9) then, by using (6.13), one can predict that the ground state energy E_0 will be

$$\begin{aligned} E_0 &\equiv U(T=0)\cdot V = U_0 V \\ &= -\frac{m\Delta E^2}{8\pi a}\cdot V = -\frac{\lambda}{2}\rho_{\mathrm{He}}^2 V\,. \end{aligned} \tag{6.23}$$

As expected, the ground state energy is negative, thus indicating that liquid Helium is a N-body bound state. In the previous calculation the zero-point motion is not taken explicitly into account. At the zero-loop level there is no contribution of the kinetic energy to the internal energy. When one takes into account quantum corrections one expects zero-point motion contributions. In fact, at the one-loop level one gets, besides the classical result, a zero-point energy given by

$$\int d^3\mathbf{k}\,\omega(\mathbf{k})$$

where $\omega(\mathbf{k})$ is given by (6.10). Since the zero-point energy contribution is infinite one performs the usual zero-point energy subtraction. One expects, in any case, quantum corrections to the zero-loop result (6.23) and these quantum corrections will give contributions to the binding energy per particle.

6.4. *Helium density*

It can be expected that in the zero temperature limit the condensate contains all the particles. This result is obtained within the dilute gas approximation. In this way one can predict that the Helium density will be

$$\rho_{\mathrm{He}} \equiv \frac{N}{V} = \frac{m\Delta E}{4\pi a}\,. \tag{6.24}$$

By using (1.2) and (6.24) one predicts

$$\rho_{\mathrm{He}} = 1.9\cdot 10^{8}\frac{\mathrm{Atom}}{(\mathrm{eV})^{-3}} = 2.45\cdot 10^{22}\frac{\mathrm{Atom}}{(\mathrm{cm})^{3}} \tag{6.25}$$

One notes at this point that result (6.25) is very close to the zero temperature Helium density (2.7):

$$\rho_{\mathrm{He}}^{\mathrm{exp}} = 1.68\cdot 10^{8}\frac{\mathrm{Atom}}{(\mathrm{eV})^{-3}} = 2.17\cdot 10^{22}\frac{\mathrm{Atom}}{(\mathrm{cm})^{3}}\,. \tag{6.26}$$

6.5. *Pressure at zero temperature*

From (6.15) one predicts that at $T = 0$ the pressure will be

$$P(0) = \frac{\Delta E^2 m}{8\pi a} = \frac{\lambda}{2}\rho_{\text{He}}^2 . \tag{6.27}$$

Furthermore, from (4.19), it follows that for temperatures smaller than the critical one the pressure is approximately constant. This prediction also finds good support from experiments.

Our prediction for $P(0)$ implies, from (1.2) and (6.27), that

$$P(0) = 85.3 \cdot 10^4 \ (\text{eV})^4 = 17 \text{ Atm} . \tag{6.28}$$

By extrapolating the isopycnicals in Keesom's book[9] we would get (for $\rho_{\text{He}}^{\text{exp}}$ given by (6.26)), the value

$$P(0) \cong 4.88 \cdot 10^4 (\text{eV})^4 = 10 \,\text{Atm} . \tag{6.29}$$

6.6. *Binding energy per particle*[2,10]

Since liquid He-4 is a N-body bound state, liquid Helium has a binding energy at absolute zero. This binding energy is equivalent to the latent heat of vaporization at $T = 0° \text{K}$. The experimental value for the Helium binding energy per atom (ε) is[2,10]

$$\varepsilon^{\text{exp}} = -13 \,\text{cal/mol} = -5.5 \cdot 10^{-4} \,\text{eV/Atom} . \tag{6.30}$$

The definition of ε is

$$\varepsilon = \frac{E_0}{N} \tag{6.31}$$

where E_0 is the ground state energy. From (6.31), (6.23) and (1.2) we get

$$\varepsilon = -\frac{\Delta E}{2} = -4.4 \cdot 10^{-4} \,\text{eV/Atom} = -10.4 \,\text{cal/mol} . \tag{6.32}$$

Our prediction (6.32) is then very close to the experimental value (6.30).

6.7. Equation of state

From (6.21) and (6.22) one can predict the equation of state at zero temperature

$$PV = -E_0 . \tag{6.33}$$

This equation is just Eq. (3.26). We have checked the validity of (2.40) up to the one loop level. If one divides Eq. (6.33) by N we get a simple relation among the physically relevant quantities,

$$\frac{P(0)}{\rho_{\mathrm{He}}} = -\varepsilon \,. \tag{6.34}$$

By using (6.29) and (6.26) we will get

$$\frac{P(0)}{\rho_{\mathrm{He}}} = 3 \cdot 10^{-4}\,\mathrm{eV} = 7.1\,\mathrm{cal/mol} \,. \tag{6.35}$$

Our prediction, from (6.28) and (6.25), is

$$\frac{P(0)}{\rho_{\mathrm{He}}} = 4.4 \cdot 10^{-4}\,\mathrm{eV} = 10.4\,\mathrm{cal/mol} \,. \tag{6.36}$$

Our predictions for the equation of state and zero-temperature pressure deserves to be better tested experimentally.

6.8. *Sound speed*

The sound speed, defined by

$$v_s = \sqrt{\frac{\partial P}{\partial \rho}} \tag{6.37}$$

can be obtained from (6.27). One gets

$$v_s = \sqrt{\frac{\lambda \rho_{\mathrm{He}}}{m}} = \sqrt{\frac{\Delta E}{m}} \,. \tag{6.38}$$

By using the values of ΔE and m in (1.2) we obtain

$$v_s = 5 \cdot 10^{-7} = 150\,\mathrm{m/s} \,. \tag{6.39}$$

6.9. *London relation*

By using (6.21) and (6.22) one can check explicitly, within the one loop level, that, as predicted on general grounds by Eq. (3.23),

$$PV + U - ST = 0 \,. \tag{6.40}$$

Furthermore if one differentiates (6.21) with respect to T and uses (6.18), one gets London's relation up to one loop level:

$$dP = \frac{S(T)}{V} dT \,. \tag{6.41}$$

6.10. *Roton spectrum*

Besides the phonon, the other type of quantum excitation, relevant in understanding the behavior of He^4 in the two fluid region, was named a roton by Landau[25,26] its discoverer. The roton dispersion relation was postulated by Landau to be of the form

$$\omega^{\text{roton}}(p) = \frac{(p-p_0)^2}{2\mu} + \Delta \tag{6.42}$$

where Δ (the energy gap), μ (the effective mass) and p_0 are phenomenological constants that could be obtained, in principle, from a microscopic theory.

The spectrum of excitations should be inferred from the solutions of the quantum mechanical many-body problem. In Feynman's approach the dispersion relation of the excitations can be inferred from the form factor for scattering of neutrons from the liquid and thus can be obtained experimentally. Improvements over Feynman's have been made very recently.[32]

Our approach to this question is that these excitations appear as transients in the symmetry restoration process. The effective Hamiltonian (4.7) describes Helium-4 at very low temperature. As the temperature increases another effective Hamiltonian might be more appropriate. However if we just admit that what happens at higher temperatures is symmetry restoration, then, by following our strategy for the relativistic case, one can get the relevant part of the spectrum at high temperatures. We recall that in the relativistic case one can get the spectrum of excitations in the symmetric phase by introducing a coupling between the low temperature phase excitations λ^+ and λ^-. By repeating this strategy in the nonrelativistic case one introduces a coupling between the Goldstone mode and the massive mode through the substitution

$$G \to G'' = \begin{pmatrix} \lambda^+(\rho_0, n, \mathbf{p}) & i\Delta E \\ i\Delta E & \lambda^-(\rho_0, n, \mathbf{p}) \end{pmatrix} . \tag{6.43}$$

The coupling between the Goldstone mode and the massive one leads to the following new modes:

$$\begin{aligned} \lambda^+(n, \mathbf{p}, \rho_0) &= \frac{\mathbf{p}^2}{2m} - \Delta E + 2\lambda\rho_0 + \sqrt{\lambda^2\rho_0^2 - \left(\frac{2n\pi}{\beta}\right)^2 - (\Delta E)^2} \\ \lambda^-(n, \mathbf{p}, \rho_0) &= \frac{\mathbf{p}^2}{2m} - \Delta E + 2\lambda\rho_0 - \sqrt{\lambda^2\rho_0^2 - \left(\frac{2n\pi}{\beta}\right)^2 - (\Delta E)^2} \end{aligned} \tag{6.44}$$

and consequently $F \to F'$ where

$$F'(\rho_0, V, t) = V\left[-\Delta E\rho_0 + \frac{\Delta\rho_0^2}{2}\right] + \frac{V}{\beta}\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ln\left(1 - e^{-\beta\omega(\mathbf{k}, \rho_0)}\right) \tag{6.45}$$

and the new dispersion relation is

$$\omega(\mathbf{k},\,\rho_0) = \sqrt{\left(\frac{\mathbf{k}^2}{2m} - \Delta E + 2\lambda\rho_0\right)^2 + (\Delta E)^2 - \lambda^2\rho_0^2}\ . \tag{6.46}$$

In order to get the thermodynamical properties of the system, and as can be inferred from Eqs. (6.5)–(6.8), all one has to do is to substitute ρ_0 by ρ_c given by (6.4). In this way, for phenomenological applications one has to analyse

$$\omega(\mathbf{k},\,\rho_c) \equiv \sqrt{\left(\frac{\mathbf{k}^2}{2m} - \Delta E + 2\lambda\rho_c\right)^2 + (\Delta E)^2 - \lambda^2\rho_c^2}\ . \tag{6.47}$$

The main prediction that can be made by using (6.47) is that the spectrum exhibits different features as the condensate becomes less populated. In particular, the spectrum (6.47) has a minimum whenever

$$\rho_c(T) < \frac{1}{2}\rho_c(T=0) \quad (2\lambda\rho_c < \Delta E)\ . \tag{6.48}$$

When condition (6.48) is met, then, from (6.47) a roton-like dispersion relation follows:

$$\omega(\mathbf{k}) \cong \sqrt{\left(\frac{\mathbf{k}^2}{2m} - \Delta E\right)^2 + (\Delta E)^2}\ . \tag{6.49}$$

For comparison we write the phonon spectrum

$$\omega^{\mathrm{ph}}(\mathbf{k}) = \sqrt{\left(\frac{\mathbf{k}^2}{2m} + \Delta E\right)^2 - (\Delta E)^2}\ . \tag{6.50}$$

The spectrum (6.47) has a minimum for

$$p_0 = \sqrt{2m\Delta E}\ . \tag{6.51}$$

At the minimum we get

$$\Delta = E(\mathrm{p}_0) = \Delta E\ . \tag{6.52}$$

The first nontrivial relation that follows from (6.51) and (6.52) is that

$$\Delta \equiv \frac{\mathrm{p}_0^2}{2m}\ . \tag{6.53}$$

Finally by expanding around $p = p_0$ one gets the roton spectrum

$$\omega^{\mathrm{r}}(p) = \frac{(p-p_0)^2}{m} + \Delta \, . \tag{6.54}$$

From (6.49) it follows that

$$\mu = \frac{1}{2} m \, . \tag{6.55}$$

In this way we have been successful in getting Landau's roton spectrum. Landau's parameters p_0, Δ and μ are predicted by our expressions (6.51), (6.52) and (6.55), by using (1.2) to be

$$\begin{aligned} &\Delta = 8.81 \cdot 10^4 \,\mathrm{eV} && \Delta_{\mathrm{exp}} = 9 \cdot 10^4 \,\mathrm{eV} \\ &p_0 = 2.6 \cdot 10^3 \,\mathrm{eV} && p_{0\,\mathrm{exp}} = 3.8 \cdot 10^3 \,\mathrm{eV} \\ &\mu \cong 1.75 \cdot 10^9 \,\mathrm{eV} && 1.12 \cdot 10^9 \,\mathrm{eV} < \mu_{\mathrm{exp}} < 2.62 \cdot 10^9 \,\mathrm{eV} \, . \end{aligned} \tag{6.56}$$

The result for the gap is in excellent agreement with the experimental result. The result for p_0 is not so good. The experimental result for μ seems to depend on the method for its determination (calorimetric method is the upper bound and neutron scattering corresponds to the lower bound).[13] In Ref. 10 it has been pointed out that both methods seems to converge to a value smaller than the lower bound.

6.11. *Superflow speed limit*

Superfluidity in Helium liquid is subject to a restriction which limits the velocity of the superfluid to values smaller than a "critical velocity". An upper bound to this critical velocity can be obtained from the knowledge of the excitation spectrum. This upper bound is[11]

$$v_{\mathrm{c}} < \left(\frac{\omega(\mathrm{p})}{p}\right)_{\mathrm{min}} \tag{6.57}$$

where $(\omega(\mathrm{p})/p)_{\mathrm{min}}$ stands for the condition

$$\frac{d}{dp}\left(\frac{\omega(\mathrm{p})}{p}\right)\bigg|_{p=p_{min}} = 0 \, . \tag{6.58}$$

The determination of the spectrum is thus important in order to get expressions for the superfluid flow velocity limit.

Our dispersion relation (6.47) can be written as

$$\omega(\mathrm{p}) = \sqrt{\left(\frac{p^2}{2m} - \frac{p_0^2(T)}{2m}\right)^2 + \Delta^2(T)} \tag{6.59}$$

where

$$p_0^2(T) = 2m[(\Delta E) - 2\lambda\rho_c(T)] \tag{6.60}$$

$$\Delta^2(T) = (\Delta E)^2 - \lambda^2\rho_c(T)\ . \tag{6.61}$$

$$v < \left(\frac{1}{m}\left[\sqrt{\left[\left(\frac{p_0^2(T)}{2m}\right)^2 + \Delta^2(T)\right]} - \frac{p_0^2(T)}{2m}\right]\right)^{1/2}\ . \tag{6.62}$$

Expression (6.62) allows us to get an upper bound for the critical body velocity. This maximum speed is

$$v < \sqrt{\frac{\max\Delta(T)}{m}} = \sqrt{\frac{\Delta E}{M}} = v_s\ . \tag{6.63}$$

The sound speed is then an upper bound to the critical velocities. It follows also from (6.63) that, if

$$\Delta(T) = 0 \tag{6.64}$$

then, there is no superfluidity.

Our conclusion is, that from (6.62), the critical velocity depends on the occupation of the condensate. The sound speed is just an upper limit. Later on we shall argue that at T_c

$$\Delta(T_c) = 0 \tag{6.65}$$

Then, from (6.64)

$$v_c(T_c) = 0 \tag{6.66}$$

This result is verified experimentally.[10]

As a final remark we would like to stress that although all results, up to Sec. 6.10 we obtained from first principles, the derivation of the roton spectrum is a rather phenomenological one. The roton spectrum was obtained as a result of the coupling between the Goldstone and the massive modes. This coupling on the other hand is introduced by analogy to the relativistic case, where this *ad hoc* procedure is known to work as a mechanism for symmetry restoration.

7. Normal and Superfluid Components

7.1. *Two-fluid picture*[14,16,17]

Below the λ-point Helium-4 behaves as a mixture of two fluids. Below 1.1° K the so-called normal component is negligible. There are many phenomenologically successful applications of the two-fluid model.[14,16] However the understanding of the two-fluid nature and a well-defined scheme for computing the thermodynamical properties of normal and/or superfluid components has not been achieved yet. In this context it is important to recall that the proposers of the two-fluid model were not in agreement with regard to the nature of the two-fluid.[17] Whereas in Landau's view the normal component is associated to the excitations, in Tisza's view the normal component was composed of Helium atoms.

By looking at our dispersion relation it is possible to shed some light on these problems. First of all one can see that for

$$p \underset{<}{\sim} p_0 = \sqrt{2m\Delta} \tag{7.1}$$

the spectrum of the excitations can be approximated by

$$\omega^{\mathrm{r}}(p) = \frac{(p-p_0)^2}{2\mu} + \Delta \,. \tag{7.2}$$

However for excitations such that

$$p > p_0 \tag{7.3}$$

the dispersion relation is of the form

$$\omega^{\mathrm{n}}(p) = \frac{\mathrm{p}^2}{2m} \,. \tag{7.4}$$

In this way one expects two types of excitations. Those whose momenta obeys (7.1) are rotons. Those obeying (7.3) will be named normal. For very low temperatures one can also distinguish these two types of excitations (although as we will see the normal component is practically absent). The first excitation will be the phonon

$$\omega(\mathrm{p}) = v_{\mathrm{s}}|p| \,. \tag{7.5}$$

For momenta such that

$$p > p_0 \tag{7.6}$$

one has

$$\omega^{\mathrm{n}}(\mathrm{p}) = \frac{\mathrm{p}^2}{2m} \,. \tag{7.7}$$

From (7.2), (7.4), (7.5) and (7.7) it follows that one has two types of excitations in the system. Normal excitations are those with low momentum, that is, those satisfying (7.4). The other excitations can be named superfluid.

From the previous distinction one can separate each physical quantities in momentum space and write, whenever integral over momenta occurs,

$$\int d^3\mathbf{p} = \int_{\mathrm{n}} d^3\mathbf{p} + \int_{\mathrm{spfl}} d^3\mathbf{p} \ . \tag{7.8}$$

where, from (5.3) or (5.6)

$$\int_{\mathrm{n}} d^3\mathbf{p} \equiv \int_{p>p_0} d^3\mathbf{p} \tag{7.9}$$

and the superfluid component is just the complement of (7.9) and should be inferred from (7.9) and (7.8).

From (7.2), (7.4), (7.5) and (7.7) we will be able to define approximations for the integrand of certain integrals. For example, if one defines

$$N(\mathbf{p}) \equiv \left(\exp\left(\frac{\omega(\mathbf{p})}{T}\right) - 1\right)^{-1} \tag{7.10}$$

then, from Eqs. (7.2)–(7.7) and (6.49) and (6.50) one can define

$$N^{\mathrm{n}}(\mathbf{p}) = \left(\exp\left(\frac{\mathbf{p}^2}{2mT}\right) - 1\right)^{-1} \tag{7.11}$$

$$N^{\mathrm{r}}(\mathbf{p}) = \left(\exp\left(\frac{(p-p_0)^2}{2\mu T} + \frac{\Delta}{T}\right) - 1\right)^{-1} \tag{7.12}$$

$$N^{\mathrm{ph}}(\mathbf{p}) = \left(\exp\left(\frac{v_s \mathbf{p}}{T}\right) - 1\right) \ . \tag{7.13}$$

From (7.11)–(7.13) one can define normal, roton and phonon thermal averages. For the kinetic energy one can define the roton contribution as

$$\langle K\rangle_{\mathrm{r}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\mathbf{p}^2}{2m} N^{\mathrm{r}}(\mathbf{p}) \ . \tag{7.14}$$

Expression (7.14) is just, in Landau's notation[11] $(\rho^{(n)})_{\mathrm{r}}$, that is,

$$(\rho^{\mathrm{n}})_{\mathrm{r}} = \frac{2m}{3T}\langle K\rangle_{\mathrm{r}} \tag{7.15}$$

whereas the average number of rotons is given by

$$\langle N\rangle_{\mathrm{r}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} N^{\mathrm{r}}(\mathbf{p}) \ . \tag{7.16}$$

7.2. Normal component of the fluid density

Our identification of the normal and superfluid excitation allows us to write explicit expressions for the normal and superfluid component of the fluid density. In order to do this we write the basic equation of condensation phenomena explicitly:

$$-\Delta E + \lambda\rho_c + \frac{2\lambda}{(2\pi)^3}\int d^3\mathrm{p}\,\frac{\left\{\left(\frac{\mathrm{p}^2}{2m}\right) - \Delta E + 2\lambda - \frac{\lambda\rho_c}{2}\right\}}{\sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E + 2\lambda\rho_c\right)^2 + (\Delta E)^2 + (\lambda\rho_c)^2}}$$

$$\times\frac{1}{\left[\exp\beta\sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E + 2\lambda\rho_c\right)^2 + (\Delta E)^2 - (\lambda\rho_c)^2} - 1\right]} = 0\,. \qquad (7.17)$$

If one divides (7.17) by λ and by using (6.9) and (6.24) and neglecting the condensate dependence in the integrals in (5.17) one can write

$$\rho_{\mathrm{He}} = \rho_c + \frac{2}{(2\pi)^3}\int d^3\mathrm{p}\,\frac{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)}{\sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2}}$$

$$\times\frac{1}{\left[\exp\beta\sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2} - 1\right]}\,. \qquad (7.18)$$

Let us decompose the integral in (7.18) into two pieces, following (7.8), and write

$$\rho_{\mathrm{He}} = \rho_s + \rho_n\,. \qquad (7.19)$$

The normal component of the Helium density should be inferred from (7.17) when one takes into account restriction (7.6). For p satisfying (7.6) the integrand of (7.18) can be approximated by

$$\frac{\frac{\mathrm{p}^2}{2m} - \Delta E}{\sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2}} \times \frac{1}{\exp\left(\frac{\varepsilon(\mathrm{p})}{T}\right) - 1} \xrightarrow[p^2 > p_0^2]{} \frac{1}{\exp\left(\frac{\mathrm{p}^2}{2mT}\right) - 1} \qquad (7.20)$$

The normal component of the superfluid density can be written, from (7.20) and (7.9) as

$$\rho_n(T) = \frac{2}{(2\pi)^3}\int_{p>p_0}\frac{d^3\mathrm{p}}{\exp\left(\frac{\mathrm{p}^2}{2m}\frac{1}{T}\right) - 1} \equiv 2\int_{p>p_0}\frac{d^3\mathrm{p}}{(2\pi)^3}\,, \quad N^{(n)}(\mathrm{p})\,. \qquad (7.21)$$

If one takes for p_0 the lower limit of (7.21) one can write a simple analytical expression for the normal component. This expression is

$$\rho_{\mathrm{n}}(T,\,\Delta E) = (2mT)^{3/2} h_0\left(\frac{\Delta E}{T}\right) \tag{7.22}$$

where

$$h_0\left(\frac{\Delta E}{T}\right) = \frac{1}{\pi^2}\int_{\sqrt{\frac{\Delta E}{T}}}^{\infty} dy\, y^2 \frac{1}{e^{y^2}-1}\ . \tag{7.23}$$

In Fig. (2) we show the behavior, as a function of T, of ρ_{n}/ρ

$$\frac{\rho_{\mathrm{n}}(T,\,\Delta E)}{\rho_{\mathrm{n}}(T_\lambda,\,\Delta E)} \equiv \left(\frac{T}{T_\lambda}\right)^{3/2} \frac{h_0\left(\frac{\Delta E}{T}\right)}{h_0\left(\frac{\Delta E}{T_\lambda}\right)}\ . \tag{7.24}$$

From Fig. 2 we can see that our expression (7.24) gives a strikingly good result from the experimental point of view. In Fig. 2 we present also the result of Andronikashvili.[33] The agreement is very good. We have verified that an equally good result is obtained if one uses instead of (7.21)

$$\rho_{\mathrm{n}}(T,\,\Delta E) = 2\int_{p>p_0} \frac{d^3\mathbf{p}}{(2\pi)^3} \times \frac{1}{\exp\left(\frac{\omega(\mathbf{p})}{T}\right) - 1} \tag{7.25}$$

with $\omega(\mathbf{p})$ given by (4.49).

One can easily see that ρ_{n} defined in (7.19) exhibits the following properties:

$$\rho_{\mathrm{n}} \xrightarrow[T\to 0]{} 0 \tag{7.26}$$

$$\rho_{\mathrm{n}} \xrightarrow[T\to\infty]{} 2\rho_{\mathrm{gas}}(T) \equiv 2\int \frac{d^3\mathbf{p}}{(2\pi)^3} \times \frac{1}{\left[e^{\frac{\mathbf{p}^2}{2mT}} - 1\right]} \tag{7.27}$$

where ρ_{gas} is the free gas density. In particular, from (7.22) and (7.23), that for $\Delta E = 0$

$$\rho_{\mathrm{n}}(\Delta E = 0,\, T) = 2\rho_{\mathrm{gas}}(T)\ . \tag{7.28}$$

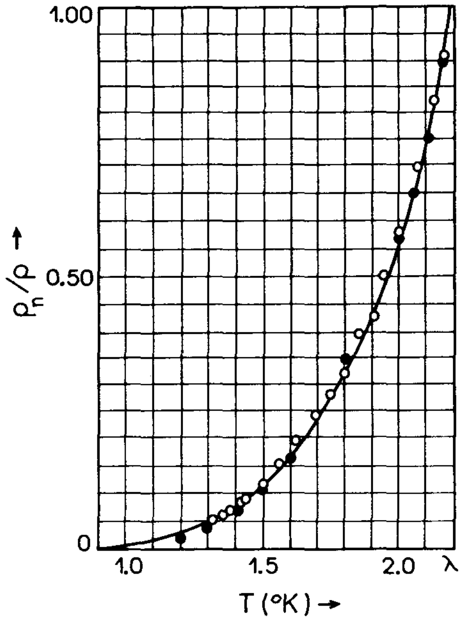

Fig. 2. ρ_n/ρ as function of temperature. Open circles are experimental values of Andronikashvili. Dark circles were obtained from $\rho_n(T)\rho(T = 2.2^\circ \text{ K})$ as proposed in this paper. Solid line represents the fit $\rho_n/\rho = (T/T_\lambda)^{5.6}$.

7.3. *Superfluid component of the fluid density*

The definition of the superfluid component is, from (7.19), given by

$$\rho_s = \rho_{\text{He}} - \rho_n \tag{7.29}$$

where ρ_n is defined by (7.21). As can be seen from (7.18) part of the superfluid component of the fluid density is made of particles that are in the condensed state (the $\mathbf{p} = 0$ state). The other part has a contribution due to excitations. In this way one has, on general grounds

$$\rho_s \cong \rho_c(T) + \rho_{\text{exc}}(T) \ . \tag{7.30}$$

The contribution due to excitations ($\rho_{\text{exc}}(T)$) depends on whether phonons or roton contributions are the relevant excitations. In the following we make a simple estimate of the roton contribution to the superfluid component as defined by equation (7.29). For the roton spectrum the relevant integration comes from the region in momentum space for which $p \cong p_0$. Let us then define from (7.18) the roton contribution to ρ_s as

$$(\rho_s)_r = 4\pi \int_0^{2p_0} \frac{p^2dp}{(2\pi)^3} \frac{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)}{\sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2}} \times \frac{1}{\left[\exp \beta \sqrt{\left(\frac{\mathrm{p}^2}{2m} - \Delta E\right)^2 + \Delta E - 1}\right]} . \tag{7.31}$$

After a change of variables (7.31) can be written under the alternative form

$$\begin{aligned}(\rho_s)_r &= \frac{1}{\pi^2} p_0^3 \int_0^2 \frac{y^2dy(y^2-1)}{\sqrt{(y^2-1)^2+1}} \times \frac{1}{\exp\left(\frac{\Delta E}{T}\sqrt{(y^2-1)^2+1}\right) - 1} \\ &= p_0^3 g\left(\frac{\Delta E}{T}\right)\end{aligned} \tag{7.32}$$

If one expands the exponential around $y = 1$ (roton contribution) and takes for the denominator its maximum value in the interval one gets the estimate for $(\rho_s)_r$:

$$\begin{aligned}(\rho_s)_r &\cong \frac{p_0^3}{3\pi^2} e^{-\frac{\Delta}{T}} \int_0^2 y^2(y^2-1)dy \\ &\cong \frac{(2m\Delta E)^{3/2}}{\pi^2} e^{-\frac{\Delta E}{T}} .\end{aligned} \tag{7.33}$$

Landau's estimates[11] for $(\rho_s)_r$ is

$$\left(\frac{N_r}{V}\right)^{\mathrm{L}} = \frac{1}{2(\pi)^{3/2}}(2m\Delta E)(mT)^{1/2} e^{-\frac{\Delta E}{T}} \tag{7.34}$$

From (7.33) and (7.34) it follows that

$$\frac{(\rho_s)_r^{\mathrm{ours}}}{\left(\frac{N_r}{V}\right)_{\mathrm{L}}} = 2\sqrt{\frac{2}{\pi} \times \frac{\Delta E}{T}} . \tag{7.35}$$

Expression (7.35) shows that our definition ρ_s is essentially identical to Landau's.

The agreement between our expression is even better when compared with Landau's $(\rho^n)_r$ defined in (7.15). In this case one gets, from (7.33) and (7.15)

$$\frac{(m\rho_s)_r^{\mathrm{ours}}}{(\rho_n)_r^{\mathrm{L}}} = \sqrt{\frac{8}{\pi}\left(\frac{T}{\Delta E}\right)} . \tag{7.36}$$

Once again the agreement with Landau's estimates is very good in the temperature interval 1.1° K $< T < 2, 1^\circ$ K.

As a final remark in this section we would like to point out that equation (7.18) leads to the following equations for the occupation of the $\mathbf{p} = 0$ state:

$$\rho_c(T) = \rho_{\mathrm{He}} - \frac{2}{(2\pi)^3}\int d^3\mathbf{p}\,\frac{d\omega(\mathbf{p}, \rho_c)}{d\left(\frac{\mathbf{p}^2}{2m}\right)} N(\mathbf{p}) \tag{7.37}$$

where ω is given by (6.46) and $N(\mathbf{p})$ is defined in (7.10).

7.4. *Critical velocity*

The main experimental evidence for our interpretation of the normal component of the fluid as being associated with excitations of the ideal gas type lies in the study of the interaction of the superfluid component with the walls. As pointed out earlier by Ginzberg,[34] although these excitations might not be very numerous, thus making a negligible contribution to the thermodynamic functions, they would be important in dissipating the flow. Let us see that for films of very narrow slits ($d < 10^{-4}$ cm) one can make a simple prediction for the critical velocity.

The dispersion of the momenta is given by the uncertainty principle as

$$\Delta p d \sim 1 \tag{7.38}$$

where d is the characteristic dimension of the film or slit. Following Ginzberg[12,34] one then has to minimize, for determining v_c the expression

$$\left(\frac{\mathbf{p}^2 + \Delta p^2}{2mp}\right)'_{\mathrm{min}} . \tag{7.39}$$

From (7.38) and (7.39) one gets

$$v_c d < \frac{1}{m}(1.6 \cdot 10^{-4}\,\mathrm{cm}^2/\mathrm{seg}) . \tag{7.40}$$

The bound (7.40) was a much searched one some years ago (see discussion in the Atkins book[12]). This bound has also been obtained by Fetter[35] and Anderson.[21] Although a dependence of the form $v_c d^{1/4} \approx 1$ is more appropriate from the phenomenological point of view,[10] some experiments exhibit the behavior predicted by (7.40).[10] The conclusion is then that the normal excitations are responsible for dissipating the flow if a certain velocity is exceeded. Furthermore it explains why the critical velocity is several orders of magnitude smaller than that predicted by the phonon or roton spectrum.

8. The λ-Point

Any attempt to describe Helium-4 should predict a transition at $T_\lambda = 2.18^\circ$ K. The determination of the critical temperature is a longstanding problem in superfluid Helium-4. In the pioneering work of London, in which he suggested that the λ-transition can be attributed to the macroscopic occupation of the $\mathbf{p} = 0$ state (Bose–Einstein condensation), he made a successful prediction for T_λ. London's result was[1,2]

$$T_c^L = 3.13^\circ \text{ K} . \tag{8.1}$$

As pointed out in Sec. 2 one of the basic ingredients of our approach to Bose–Einstein condensation, is the determination of the occupation of the macroscopic state with $\mathbf{p} = 0$. The critical temperature is the one for which, from Eq. (3.22), this occupation is zero. By using this criterion, one gets the following equation for determining the critical temperature:

$$T_c \frac{\partial}{\partial \rho_o}\left[\int \frac{d^3\mathbf{p}}{(2\pi)^3} \ln\left(1 - e^{-\frac{\omega(\mathbf{p},\rho_0)}{T_c}}\right)\right]_{\rho_0 = 0} = \Delta E . \tag{8.2}$$

The critical temperature equation (8.2) can be written under a fairly simple form if one uses (7.10) and (6.24):

$$2\int \frac{d^2\mathbf{p}}{(2\pi)^3} \frac{d\varepsilon(\mathbf{p}, \rho_0 = 0)}{d\left(\frac{\mathbf{p}^2}{2m}\right)} N(\mathbf{p}) = \rho_{\text{He}} \tag{8.3}$$

where $N(\mathbf{p})$ is defined in (7.10). Clearly equation (8.3) can also be obtained from (7.34) by imposing $\rho_c(T_c) = 0$.

By using the dispersion relation in (6.49) in (8.2) or, equivalently, (8.3), one gets the following expression for T_c:

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\left(\frac{\mathbf{p}^2}{2m} - \Delta E\right)}{\sqrt{\left(\frac{\mathbf{p}^2}{2m} - \Delta E\right)^2 + \Delta E^2}\left[\exp \beta_c \sqrt{\left(\left(\frac{\mathbf{p}^2}{2m}\right)^2 - \Delta E\right)^2 + (\Delta E)^2} - 1\right]} = \frac{m\Delta E}{8\pi a} . \tag{8.4}$$

After some changes of variables, expression (8.4) can be written as

$$f\left(\frac{\Delta E}{T_c}\right) = \frac{\pi}{16\sqrt{2}a\sqrt{m\Delta E}} \tag{8.5}$$

where

$$f\left(\frac{\Delta E}{T_c}\right)=\int_0^\infty \frac{dyy^2(y^2-1)}{\sqrt{(y^2-1)^2+1}}\frac{1}{\left\{\exp\left[\frac{\Delta E}{T_c}\sqrt{(y^2-1)^2+1}\right]-1\right\}}\,. \tag{8.6}$$

A numerical solution of Eq. (8.5) leads us to

$$T_c = 5.0^\circ\,\mathrm{K}\,. \tag{8.7}$$

Although result (8.7) is a factor 2 of the experimental result it shows clearly that Bose–Einstein condensation is indeed related to the λ-point. If one is able to understand what is happening at the phase transition one would get a better result for the critical temperature.

We have shown that at zero temperature the particles in the condensate are bound. Let us analyze under which conditions there would be unbinding of particles as a result of temperature increase. This would happen for T_c given by

$$\frac{3}{2}T_c = \frac{\Delta}{2}\,. \tag{8.8}$$

Condition (8.8) implies

$$T_c = 2.93^\circ\,\mathrm{K}\,. \tag{8.9}$$

If our argument that the critical temperature is the one for which there is unbinding of the atoms in the condensate, then a better estimate of the critical temperature would be

$$N\frac{\Delta}{2} = V\int d^3\mathbf{p}\frac{\mathbf{p}^2}{2m}N(\mathbf{p})\,; \tag{8.10}$$

that is, when the average kinetic energy is equal to the total energy of the bound system

$$\rho_{\mathrm{He}}\left(\frac{\Delta}{2}\right) = \int d^3\mathbf{p}\frac{\mathbf{p}^2}{2m}N(\mathbf{p})\,. \tag{8.11}$$

One takes into account only the roton contribution to (8.11) and writes

$$\rho_{\mathrm{He}}\left(\frac{\Delta}{2}\right) \cong \int d^3\mathbf{p}\frac{\mathbf{p}^2}{2m}N_r(\mathbf{p})\,. \tag{8.12}$$

with N_r given by (5.12). Expression (8.12) is just a variant of Landau's condition $(\rho_n)_r = \rho_{He}$ which implies

$$\rho_{He}\left(\frac{3T}{2}\right) = \int d^3p \frac{p^2}{2m} N_r \, . \tag{8.13}$$

The idea that at the critical temperature there is an unbinding of the N-body bound state leads us to the determination of the critical temperature if one imposes further that at the critical temperature

$$\Delta E(T_c) = 0 \, . \tag{8.14}$$

The requirement (8.14) is equivalent to saying that the binding energy at the critical temperature is zero. Formally condition (8.14) is equivalent, within the field theoretical context, to the condition

$$\rho_c(T_c) = 0 \tag{8.15}$$

since condition (8.14) only requires that the linear coefficient of the expansion of the free energy as a function of the condensate vanishes. We should then analyse better this question by using a field theory approach. For the moment we just note that if one imposes that the gap is zero then equation (8.3) can be written as

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left(\dfrac{p^2}{2mT_c}\right) - 1} = \frac{\rho_{He}}{2} \equiv \rho_{gas}(T_c) \, . \tag{8.16}$$

The striking feature of equation (8.16) is its similarity to the critical temperature equation derived by London within the free Bose gas approximation. The critical temperature determination by London T_c^L was

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{\exp\left(\dfrac{p^2}{2mT_c^L}\right) - 1} = \rho_{He} \equiv \rho_{gas}(T_c^L) \, . \tag{8.17}$$

From (8.17) and (8.16) it follows that

$$\frac{T_c^{ours}}{T_c^L} = 2^{-2/3} \, . \tag{8.18}$$

From (8.18) and (8.1)

$$T_c = 2.0^\circ \text{ K} \tag{8.19}$$

follows.

Clearly result (8.16) could be inferred directly from (8.3) by imposing that when there is no Bose–Einstein condensation the dispersion relation is the one of free particles; that is, if one assumes that for (8.3) only normal particles contribute we get (8.16).

Finally we can see that for the critical temperature determined from condition (8.14) it follows from (7.18) that the critical temperature is the one for which

$$\rho_{\rm n}(T_{\rm c}) = \rho_{\rm He}\ . \tag{8.20}$$

Result (8.20) is verified experimentally, and it follows from (7.24) and (8.16).

So far we have succeeded in estimating the point in which the transition occurs (the critical temperature). A complete description of the λ-transition requires further information which we have not provided in this paper. A complete description would include the calculation of the discontinuity of the specific heat at the λ-point and the calculation of other thermodynamic variables — such as the entropy — close to the λ-point. Presumably such a complete description is bound to fail if one restricts ourselves to the one-loop approximation. The analysis of the order of the phase transition shows that quantitatively there are differences between the λ-transition and Bose–Einstein condensation predicted by the semiclassical method. The former is second order while the latter is first order. The reason for the failure of any semiclassical method or any scheme that treats $b(0)$ ($b^+(0)$) as a c-number is that, as the critical point is approached, these variables no longer can be treated as classical variables. One has to look for more reliable schemes to deal with some other predictions for the λ-transition.

In view of the above comment the result obtained by us, $T_\lambda \cong 2°$ K, is somehow surprising. Within the scheme presented in Ref. 7 one has a well-defined prescription for computing the critical temperature in field theory. The one loop approximation is a systematic and simple computational method in field theory. With a computational method in hand one has to try to get as much information as we can. In the case of the critical temperature the result we got is better than expected.

A surprising result obtained by us is the similarity between our condition (8.16) and the one obtained by London using the ideal gas model (8.17). In our approach we deal with an interacting gas but we end up with a condition that differs from London's by only a factor of two. In spite of the similarity, our result shows that the λ-transition is not strictly that of an ideal gas as London assumed.

We leave to a future publication a better description of the λ-point (computation of the specific heat, viscosity coefficient and the heat conductivity) as well as a derivation of $T_{\rm c}$ using the field theoretical approach to the gap equation.[36]

9. Conclusions

In this paper we have shown how the understanding of many features of the superfluidity phenomenon can be achieved from first principles. The model proposed here has no free parameters and the two parameters in the model (ΔE and a) can be inferred from the Helium–Helium Interaction potential. The justification for using few parameters in the description of this quantum liquid is well-known[4,5] and relies on the fact that, at low temperatures, imperfect gases are insensitive to details of the interatomic potential.

In order to make specific predictions, using the Field theoretical method, we have used the method proposed in Ref. 7 and the semiclassical approximation. For the description of the low temperature physics we carried out a complete semiclassical approximation. This method is quite standard in Field Theory and its justification for treating Bose–Einstein condensation can be traced back to the pioneering work of Bogoliubov. The results obtained in this paper, by employing the so-called one loop approximation, makes the difficult problem of estimating higher loops' contributions a very interesting and relevant one.

We achieve, by using the effective interaction Hamiltonian (4.7), a good description of superfluid Helium for $T < 1.0^\circ$ K. This is the range of validity of the low temperature effective Hamiltonian (4.7). Although we have not succeeded in writing an effective Hamiltonian for $T \gtrsim 1.0$ we have succeeded in getting the roton spectrum by arguing that the system is in the process of restoring the symmetry. By using the relativistic prescription we have been able to get the roton spectrum.

Another result worth emphasizing is the identification of the normal component of the fluid as Helium atoms whose momentum is larger than $\sqrt{2m\Delta E}$. This confirms a conjecture first made by Tisza[16] on the nature of the normal fluid (this is his postulate (c)). Such a finding allows us to write explicit expressions for all thermodynamical variables involving the normal component. The expression for the normal component of the fluid density is

$$\frac{\rho_n(T)}{\rho_n(T_c)} = \left(\frac{T}{T_\lambda}\right)^{3/2} \frac{h\left(\frac{\Delta E}{T}\right)}{h\left(\frac{\Delta E}{T_c}\right)} .$$

where h is defined in (5.23). The dependence of $\rho_n(T)/\rho_n(T_c)$ exhibited in Fig. 2 is in good agreement with Andronikashisli's experimental determination of $\rho_n(T)/\rho$.

Finally we comment on the determination of the critical temperature. In our first estimate we have taken the gap as independent of temperature. This is obviously true in the "classical approximation". One expects that as a result of quantum thermal fluctuations the gap disappears. Within the constant gap approximation we get $T_c = 5.0^\circ$ K. However if one imposes in some of our equations that at the critical temperature both the condensate and the gap vanishes

$$\rho_c(T_c) = 0$$

$$\Delta E(T_c) = 0 .$$

Then, from (6.15), it follows that

$$T_c = \frac{2\pi}{m}\left[\zeta^{-1}(3/2)\frac{\rho_{\rm He}}{2}\right]^{2/3} \cong 2^\circ \ {\rm K}\ .$$

Furthermore, from (5.22) and (5.18) we get

$$\rho_{\rm n}(T_c) = \rho_{\rm He}\ .$$

Since, on the other hand, (5.24) predicts

$$\frac{\rho_{\rm n}}{\rho}(T \sim 1.1) \cong 0$$

one gets results in agreement with the experiments. The above results are just Tisza's postulates (c) and (d). In terms of the three fundamental parameters a, ΔE and m one can write for the critical temperature

$$T_c = \frac{2\pi}{m}\left(\frac{m\Delta E}{8\pi a}\frac{1}{\zeta(3/2)}\right)^{2/3}$$

whereas for the normal component we get

$$\frac{\rho_{\rm n}(T)}{\rho_{\rm He}} = \left(\frac{T}{T_\lambda}\right)^{3/2} \frac{\displaystyle\int_{\sqrt{\frac{\Delta E}{T}}} dy y^2 \frac{1}{e^{y^2}-1}}{\displaystyle\int_2^\infty dy y^2 \frac{1}{e^{y^2}-1}}\ .$$

One of the byproducts of our paper was to shed some light on the Landau–Tisza controversy.[17] In fact, from (5.18) it follows from Helium density is composed of a uniform background (Bogoliubov's condensate) plus excitations. However, some excitations are just Helium atoms. We have shown that the normal component is associated to these atoms. With regard to the separation of the Helium density we found results that are compatible with Landau's view. However, as far as the identification of the normal component is concerned we found that Tisza's view is more appropriate.

Our approach to Bose–Einstein condensate makes a simple prediction for the equation of state at zero temperature:

$$\frac{P}{\rho} = -\epsilon$$

which deserves to be tested experimentally.

There are two relevant achievements in our paper. First, we get, in our description, nine features of superfluidity. These features constitute a test for any microscopic theory. Clearly some of our predictions deserve further analysis. For example, although we predict the condensate, we have not studied the depletion of

Table 1. Basic features of superfluid Helium.

Specific feature	Section	Prediction (our model)	Experimental
phonons	6.1	predicted	observed
N-Body Bound State	6.3	$E_0 = -\frac{\lambda}{2}\rho_{\mathrm{He}}^2 V$	see E_0/N in Table 2
Gibbs Free Energy	2.4	$\mu = U + PV - ST = 0$	agreement with London's relation
rotons	6.10	predicted	observed
two-fluid	7.1	predicted	good agreement for ρ_n/ρ
thermodynamics	6.2	predictions for P, S, C_v	good agreement using phonon spectrum
Bogoliubov's condensate	6 and 7	predicted	observed
critical velocity	6.11	depends on the condensate. $v_c(T_c) = 0$	$v_c(T_c) = 0$
broken symmetry aspect	4.1 conclusion	$U(1)$ symmetry is spontaneously broken	existence of the condensate

the $\mathbf{p} = 0$ state. This is important in order to make comparisons with experiments. The nine features of superfluidity are presented in Table 1.

The second important achievement is the determination of twelve relevant physical quantities by using the three parameters defined in (1.2). The results are presented in Table 2. We get good agreement for London's relation, ρ_n/ρ and the energy gap. The result for T_λ is in a fact, the best estimate of Sec. 6. The agreement for other physical quantities can be inferred by looking at Table 2.

Our conclusion is that we succeeded in getting, from a microscopic theory, not only the essential features of Helium superfluid but we have succeeded in making quantitative predictions for a large number of macroscopic parameters of superfluid Helium.

Much work will be required in order to achieve a complete description of superfluidity. First of all one has to compute the gap equation field theory[36] (in order to get another determination of the critical temperature) and to study the depletion of the $\mathbf{p} = 0$ state. The next step should be the computation transport coefficients (such as the viscosity). Furthermore one has to get a more complete description of the λ-transition (including the behavior of the specific heat). Presumably such a complete description will require a method different from the loop-expansion (employed in this paper). The estimates of higher order terms in the loop-expansion will also be relevant.

Table 2. Specific numerical predictions.

physical quality	Section	expression	predicted	experimental
Helium density	6.4	$\rho_{\mathrm{He}} = \frac{m\Delta E}{4\pi a}$	$2.45 \cdot 10^{22}$ Atom/cm^3	$2.17 \cdot 10^{22}$ Atom/cm^3
binding energy per Atom	6.6	$\epsilon = -\frac{\Delta E}{2}$	-10.4 cal/mol	-13 cal/mol
sound speed	6.8	$v_s = \sqrt{\frac{\Delta E}{m}}$	150 m/s	210 m/s
Landau's parameter p_0	6.10	$p_0 = \sqrt{2m\Delta E}$	$2.6 \cdot 10^3$ eV	$3.8 \cdot 10^3$ eV
energy gap	6.10	$\Delta = \Delta E$	$8.81 \cdot 10^{-4}$ eV	$9 \cdot 10^{-4}$ eV
Landau's effective mass	6.10	$\mu = \frac{m}{2}$	$\mu = \frac{m}{2}$	$0.3\,\mathrm{m} < \mu < 0.7\,\mathrm{m}$
pressure	6.5	$P(0) = \frac{m\Delta E^2}{8\pi a}$	$8.3 \cdot 10^4\ (\mathrm{eV})^4$ 17 Atm	$4.8 \cdot 10^4\ (\mathrm{eV})^4$ 10 Atm
critical velocity	7.4	$v_c d < \frac{1}{m}$	$< \frac{1}{3.5} 10^{-9}\ \mathrm{eV}^{-1}$	$< \frac{1}{3.5} 10^{-9}$ eV
Equation of state	3.3	$PV = -E_0$	$\frac{P}{\rho_{\mathrm{He}}} = 4.4 \cdot 10^{-4}$ eV 10.4 cal/mol	$\frac{P}{\rho_{\mathrm{He}}} = 3 \cdot 10^{-4}$ eV 7.1 cal/mol
critical temperature	8.1	$T_\lambda \simeq \frac{2\pi}{m}\left[\frac{m\Delta E}{8\pi a\zeta(3/2)}\right]^{2/3}$	2° K	2.18° K
normal component	7.2	$\frac{\rho^n}{\rho} \simeq \left(\frac{T}{T_\lambda}\right)^{3/2} \frac{h\left(\frac{\Delta E}{T}\right)}{h(4)}$	see Fig. 2	excellent agreement

References

1. F. London, *Phys. Rev.* **54**, 947 (1938).
2. F. London, *Superfluids*, Vol. 2 (John Wiley, New York, 1954).
3. P. Nozières and D. Pines, *The Theory of Quantum Liquids — Superfluid Bose Liquids* (Addison-Wesley, New York, 1990).
4. K. Huang, *Statistical Mechanics* (Wiley, New York, 1963).
5. R. K. Pathria, *Statistical Mechanics* (Pergamon, Oxford, 1978).
6. J. E. Kilpatrik, W. E. Keller and E. F. Hammel, *Phys. Rev.* **97**, 9 (1955).
7. G. C. Marques, *Phys. Rev.* **B44**, 12485 (1991).
8. N. N. Bogoliubov, *J. Phys.* **11**, 23 (1947).

9. M. H. Keesom, *Helium* (Elsevier, London, 1942).
10. W. E. Keller, *Helium-3 and Helium-4* (Plenum, New York, 1969).
11. L. Landau and E. Lifshitz, *Statistical Physics* (Addison Wesley, Reading, 1959), Chap. 6.
12. K. R. Atkins, *Liquid Helium* (Univ. Press Cambridge, 1959).
13. C. T. Lane, *Superfluid Physics* (McGraw-Hill, New York, 1962).
14. L. D. Landau, *J. Phys. USSR* **5**, 71 (1941).
15. R. P. Feynman, *Phys. Rev.* **91**, 1291, 1301 (1953); **94**, 262 (1954); R. P. Feynman and M. Cohen, *Phys. Rev.* **102**, 1189 (1956).
16. L. Tisza, *Phys. Rev.* **72**, 838 (1947).
17. L. D. Landau, *Phys. Rev.* **75**, 884 (1949); L. Tisza, *Phys. Rev.* **75**, 890 (1949).
18. A. A. Abrikosov, L. P. Gor'kov and I. Ye. Dzyaloshinskii, *Quantum Field Theoretical Methods in Statistical Physics* (Pergamon, Oxford, 1965).
19. F. E. Wiegel and J. B. Jalickee, *Physica* **57**, 317 (1972); F. E. Wiegel, *Phys. Rep.* **16C**, 57 (1975).
20. A. C. Olinto, *Phys. Rev.* **B43**, 7737 (1991); **B33**, 1849 (1986); **B34**, 6159 (1986).
21. P. W. Anderson, *Rev. Mod. Phys.* **38**, 298 (1966).
22. S. T. Beliaev, *Sov. Phys. JETP* **7**, 289 (1958).
23. O. Penrose and L. Onsager, *Phys. Rev.* **104**, 576 (1956).
24. H. E. Glassgold, A. N. Kaufman and K. M. Watson, *Phys. Rev.* **120**, 660 (1960).
25. M. Girardeau, *Phys. Fluids* **5**, 1468 (1962).
26. T. D. Lee, K. Huang and C. N. Yang, *Phys. Rev.* **106**, 1135 (1957).
27. T. D. Lee and C. N. Yang, *Phys. Rev.* **113**, 1165 (1959); **116**, 25 (1959); **117**, 12 (1960); **117**, 22 (1960); **117**, 897 (1960).
28. L. Michel, *Rev. Mod. Phys.* **52**, 617 (1980).
29. T. Toyoda, *Phys. Lett.* **87a**, 91 (1981); T. Toyoda, *Ann. Phys.* **141**, 154 (1982) and T. Toyoda, *Ann. Phys.* **147**, 244 (1983).
30. N. M. Hugenholtz and D. Pines, *Phys. Rev.* **116**, 489 (1959); S. T. Beliaev, *Sov. Phys. JETP* **7**, 299 (1958).
31. J. Goldstone, *Nuovo Cimento* **19**, 154 (1961).
32. W. Wu, S. Vitiello, L. Reatto and M. H. Kalos, *Phys. Rev. Lett.* **67**, 1446 (1991).
33. E. L. Andronikashvili and Yu. G. Mamaladze, *Rev. Mod. Phys.* **38**, 567 (1966); E. L. Andronikashvili, *J. Phys. USSR* **10**, 201 (1946).
34. V. L. Ginzberg, *Dokl. Akad. Nauk, USSR* **69**, 161 (1949).
35. A. L. Fetter, *Phys. Rev.* **138A**, 429 (1965).
36. G. C. Marques and R. O. Ramos, *Microscopic Theory of Superfluid Helium-4: Further Results*, in preparation.