Localization and interactions in a disordered electron gas

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We consider a model of N species of electrons in a random potential interacting via a short-range repulsive interaction. We study the $N=\infty$ limit and the $1/N$ expansion to the leading order in $1/N$. After renormalizing the theory, we find that there are three coupling constants in this problem: (i) a coupling constant with the dimensions of the resistivity, (ii) the coupling for electron-electron scattering, and (iii) the coupling strength between diffusive modes and density fluctuations. The renormalization-group equations are presented. In $2+\epsilon$ dimensions the Anderson fixed point of the noninteracting theory is shown to belong to a line of unstable fixed points. A new ("interacting") fixed point is found. At the transition we find that, to leading order in $1/N$, (a) the exponent ν of the localization length is the same as in the noninteracting theory, (b) the dc conductivity vanishes at the mobility edge with an exponent $s=\frac{2}{17}$, (c) the density of states at the Fermi surface vanishes at the mobility edge with an exponent $\delta=\frac{2}{17}$, (d) the mean free time τ at the Fermi surface vanishes at the mobility edge with an exponent $\zeta=\frac{7}{17}$, (e) the Fermi velocity diverges at the mobility edge with an exponent $\rho=\frac{5}{17}$, and (f) the diffusive modes acquire wave-function renormalization and the anomalous dimension η is (to leading order) equal to $\epsilon/34$.

I. INTRODUCTION

The physics of disordered metals is a difficult problem that has attracted great attention in the past years. The problem of localization of states in a noninteracting electronic system was first studied by Anderson.[1] Using scaling ideas, it has been realized by Abrahams *et al.*[2] that a two-dimensional dirty "conductor" cannot sustain a static conductivity no matter how weak the disorder is. Thus the lower critical dimension for localization is 2. This result follows from the infrared (logarithmic) divergent nature of the maximally crossed graphs of Langer and Neal[3] in two dimensions.[2]

Altshuler *et al.*[4] have recently found that an electron's correlation effects can also lead to logarithmic divergencies in physical quantities in two dimensions. In particular, they considered a subset of diagrams, not of the Langer and Neal form, which also lead to logarithmically divergent contributions to the conductivity. Furthermore, Altshuler *et al.* calculated corrections to the density of states and found a logarithmically divergent contribution at the Fermi surface.[4]

A scaling study in three dimensions which combines both localization and correlation effects has been performed by McMillan.[5] He suggests the existence of a nontrivial "interacting" fixed point which governs the critical behavior of a system with long-range (Coulomb) forces. He argues that interaction effects are relevant near the Anderson fixed point and hence the pure Anderson transition would not be observable. He further finds a vanishing of the density of states (at the Fermi surface) at the transition point and incomplete screening.

The purpose of this paper is to perform a systematic renormalization-group (RG) study of the critical behavior of an interacting disordered electron gas. We consider the somewhat simpler problem of short-range interactions. (The problem of long-range interactions will be the subject of a future publication.) In order to have a renormalization-group approach, a consistent perturbative scheme is needed. In previous studies, like in the work of Langer and Neal,[3] the logarithmically divergent terms were found to be of the order of $1/(k_F l)$ (l being the mean free path and k_F the Fermi momentum).

An alternative approach to the localization problem (noninteracting) has been considered by Wegner.[6] He studied an Anderson model with N orbitals per site in the limit $N\to\infty$. Oppermann and Wegner[7] considered the $1/N$ expansion of this model. They found that at $N=\infty$ the system is just a weakly disordered metal and no localized states are found. To the order $1/N$ the equivalent of the cross-ladder graphs of Langer and Neal[3] are generated. The $1/N$ expansion thus organizes the contributions by their degree of infrared divergence.

In this paper we generalize this model to include interactions and perform a $1/N$ expansion. Oppermann[8] has considered a similar model. The advantages of the $1/N$ expansion are the following.

(i) It gets the right limits at $N=\infty$. It reproduces the results of the Hartree approximation (for the one-particle Green's functions) and the random-phase approximation

(RPA) (for the two-particle Green's function) in the zero-disorder limit. Also, as shown by Wegner,[6] it reduces to the coherent-potential approximation (CPA) in the noninteracting limit. The one-particle Green's functions are damped and the mean free path is finite.

(ii) When interactions are present, the $1/N$ expansion also organizes the diagrams by their degree of infrared divergence, thus providing a natural expansion parameter. A further advantage of this model is that conservation laws and the Ward identities that follow from them are automatically satisfied order by order in $1/N$.

The $1/N$ expansion is found to have infrared-divergent contributions to every order in $1/N$. The theory has to be renormalized in order to control these divergencies. We find that, in addition to the renormalization of the diffusion constant, it is also necessary to renormalize (a) the Fermi fields, (b) the mean free time, (c) the Fermi velocity, and (d) the "wave function" of the diffusive modes. Using the renormalization group an invariant theory is built and a $2+\epsilon$ expansion is obtained.

In this system there is a set of three parameters which determine the critical behavior:

(a) A parameter x,

$$x=\frac{1}{2\pi\rho D}\ , \tag{1.1}$$

proportional to the inverse of the coefficient of q^2 in the diffusive propagator. While x has dimensions of resistance, it is not the macroscopic resistance. In fact, the macroscopic resistance is not coupled to a local operator in the interacting theory.

(b) The coupling constant y between electrons (or holes) and the density fluctuations,

$$y=\frac{\rho g_{++}}{\lambda_0^2}=\frac{\rho g_{--}}{\lambda_0^2}\ . \tag{1.2}$$

(c) The coupling constant z between diffusive modes and density fluctuations,

$$z=\frac{\rho g_{+-}}{\lambda_0^2}=\frac{\rho g_{-+}}{\lambda_0^2}\ , \tag{1.3}$$

where ρ is the bare density of states at the Fermi surface (at $N=\infty$), D is the "diffusion constant" (i.e., the coefficient of q^2 in the diffusive propagator), λ_0 is the inverse interaction range, and g_{++}, g_{+-}, g_{-+}, and g_{--} are the coupling constants. In the unperturbed theory the coupling constants (1.2) and (1.3) coincide. Fluctuation effects reveal their distinct physical role.

Dimensionally, x scales like $E_c^{-\epsilon/2}$ ($\epsilon=d-2$) while y and z are dimensionless in all dimensions, where E_c is a bandwidth cutoff. The logarithmic infrared divergencies in $d=2$ can be traced back to the dimensionless character of x in $d=2$. The dimensionless nature of y and z produces, as we will see below, marginal instability of the Anderson fixed point in some direction.

We briefly summarize here the main results of this paper:

(i) In $2+\epsilon$ dimensions the phase diagram has, at least, two phases: (a) a disordered interacting metallic phase and (b) an interacting insulating phase. In two dimensions only the insulating phase is present and the system is asymptotically free at short distances.

(ii) The Anderson fixed point is shown to belong to a line of unstable fixed points. A new "interacting" fixed point is found.

(iii) The exponent ν of the localization length is found to be the equal to $1/\epsilon$ (to order $1/N$) as in the noninteracting theory.

(iv) The exponent s of the dc conductivity is $\frac{2}{17}$.

(v) The density of states at the Fermi surface is found to vanish, at the mobility edge, with an exponent $\delta=\frac{2}{17}$. The density of states near the Fermi surface at the mobility edge is found to behave like $|E/E^*|^{\theta}$ with $\theta=\epsilon/17$.

(vi) The mean free time τ acquires singular behavior near the edge. We find that τ, at the edge and near the Fermi surface, vanishes like $|E/E^*|^{\lambda}$ with $\lambda=7\epsilon/34$. Near the mobility edge, at the Fermi surface, we find that τ vanishes with an exponent $\zeta=\frac{7}{17}$.

(vii) The (Fermi) velocity, at the Fermi surface, diverges as the mobility edge is crossed with an exponent $\rho=\frac{5}{17}$ while the velocity near the Fermi surface diverges at the edge like $|E/E^*|^{-\sigma}$, where $\sigma=\frac{5}{34}\epsilon$.

(viii) The diffusive modes acquire a nontrivial anomalous dimension exponent $\eta=\epsilon/34$. In terms of η the diffusive modes behave, at zero momentum, like

$$K_B^{-1}(\omega)\sim|\omega/\omega^*|^{1-\eta}K_B^{-1}(\omega^*)\ .$$

(ix) An energy scale E_0 is found to exist in the insulating phase. We argue that this scale controls the crossover from critical-to-localized behavior. For $d>2$, E_0 diverges near the transition with an exponent equal to $2/\epsilon$. In two dimensions E_0 diverges with an essential singularity as dictated by asymptotic freedom.

The paper is organized as follows. In Sec. II we present the model and solve its $N=\infty$ limit. The $1/N$ expansion is developed in Sec. III. In Sec. IV the theory is renormalized. The renormalization-group equations are derived in Sec. V where we present the phase diagram and the scaling properties of the system are found. Section VI is devoted to the conclusions.

II. THE MODEL AND ITS $N=\infty$ LIMIT

A. The model

We consider a model system of a disordered electron gas with N species of electrons interacting via short-range, instantaneous repulsive interactions. The Hamiltonian is (in d-space dimensions)

$$H=\int d^dx\left[\psi_\alpha^\dagger(\vec{x})\left[-\frac{1}{2m}\nabla^2\right]\psi_\alpha(\vec{x})+\frac{f_{\alpha\beta}(\vec{x})}{\sqrt{N}}\psi_\alpha^\dagger(\vec{x})\psi_\beta(\vec{x})\right]$$
$$+\int d^dx\int d^dy\frac{g}{2N}\psi_\alpha^\dagger(\vec{x})\psi_\beta^\dagger(\vec{y})U(\vec{x}-\vec{y})\psi_\beta(\vec{y})\psi_\alpha(\vec{x})\ ,$$
$$\alpha,\beta=1,\ldots,N \tag{2.1}$$

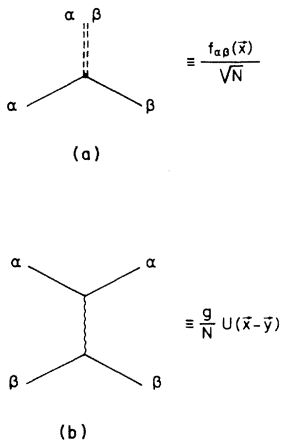

FIG. 1. (a) Elementary impurity scattering process. (b) Elementary electron-electron interaction.

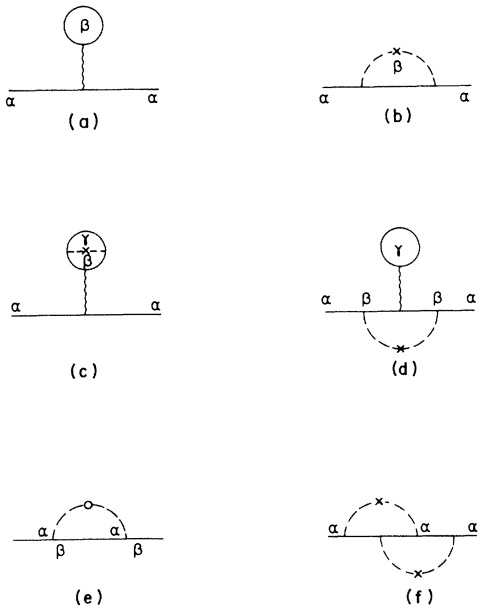

FIG. 3. Typical low-order contributions to the one-particle Green's function.

(repeated indices are summed). In formula (2.1) $\psi_\alpha(\vec{x})$ stands for the electronic Fermi field and α labels the species. The local random potential $f_{\alpha\beta}(\vec{x})$ is a random Gaussian variable with

$$\langle f_{\alpha\beta}(\vec{x}) f_{\alpha'\beta'}(\vec{x}')\rangle = \delta^{(d)}(\vec{x}-\vec{x}')M(\delta_{\alpha\alpha'}\delta_{\beta\beta'}+\delta_{\alpha\beta'}\delta_{\beta\alpha'}) ,$$
$$\langle f_{\alpha\beta}(\vec{x})\rangle = 0 . \tag{2.2}$$

Note that a factor of $1/\sqrt{N}$ has been pulled out in order to make the $N\to\infty$ limit finite. This will be discussed below. The interaction coupling constant is given by g (note the factor of $1/N$) and the short-range potential $U(\vec{x})$ is equal to

$$U(\vec{x}) = \int \frac{d^d p}{(2\pi)^d} \frac{e^{i\vec{p}\cdot\vec{x}}}{p^2+\lambda_0^2} , \tag{2.3}$$

where λ_0^{-1} is the range of the interaction. The Green's functions of this system can be calculated in a double expansion in powers of the coupling constant g and the width of the distribution M. We are interested in averages of Green's functions. They are obtained by averaging each Feynman graph independently.[9,10]

The basic processes are shown in Figs. 1(a) and 1(b). The electrons may scatter elastically off the impurities [Fig. 1(a)] or off each other [Fig. 1(b)]. Impurity averaging leads to an effective scattering between electrons shown in Figs. 2(a) and 2(b). Note that these two processes have in principle different strengths. However, in this model the scattering matrix elements $f_{\alpha\beta}(\vec{x})$ are taken to be real and the Hermiticity of the Hamiltonian implies that both processes have the same weight M/N. For a more general case see the work of Wegner,[6] Oppermann and Wegner,[7] and Oppermann.[8]

A typical set of low-order diagrams for the one-particle Green's function is shown in Figs. 3(a)–3(f). Diagrams 3(a) and 3(b) are the standard low-order self-energy correction to the one-particle Green's function. Figures 3(c) and 3(d) show two corrections to the self-energy in which both disorder and interactions are present.

B. The $N\to\infty$ limit

Let us consider first the graphs without interaction insertions and in particular let us focus on their N dependence. As discussed by Wegner,[6] graph 3(d) is of the order of $(M/N)N\equiv M$, since there is an internal sum over the index α. In contrast, diagram 3(e) has no internal sum since the indices α and β must be equal because $G_0^{\alpha\beta}(x,y)=G_0(x,y)\delta^{\alpha\beta}$. This diagram 3(e) is of the order of M/N. Diagram 3(f) is of the order of $(M/N)^2$.

The "rainbow graph" (Fig. 4), however, which contains k crosses, is of the order of $(M/N)^k N^k \equiv M^k$. Thus at $N=\infty$ the one-particle Green's functions, in the noninteracting case, contain all possible rainbows. In the opposite limit, $f_{\alpha\beta}(\vec{x})=0$ (no disorder), the $N=\infty$ limit is equivalent to the Hartree approximation to the one-particle Green's functions. Every internal loop (in the absence of disorder) carries a factor of N and energy interaction line a factor of $(g/N)U(\vec{x},\vec{y})$. Thus at $N=\infty$ the

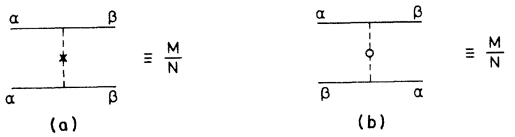

FIG. 2. Two processes involved in impurity averaging.

FIG. 4. Rainbow graph.

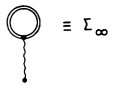

FIG. 5. Self-energy Σ_∞.

only surviving graphs are those with an arbitrary number of tadpole insertions.

1. One-particle Green's function

Let us consider now the general case $g\neq 0$, $M\neq 0$. At $N=\infty$ the average one-particle Green's function, in the noninteracting limit, satisfies the Dyson equation,

$$G_\infty(x,y)=G_0(x,y)+M\int dz\, G_0(x_1,z)G_\infty(z,z)G_\infty(z,y)\ , \tag{2.4}$$

where we have dropped the species indices since the average G's are diagonal. In Fourier space (2.4) leads to a one-particle Green's function equal to[9,6]

$$G_\infty^{-1}(p,\omega)=\omega-E_0(\vec{\mathrm{p}})+\mu+i\frac{\mathrm{sgn}\omega}{2\tau}\ , \tag{2.5}$$

where the mean free time τ is given by

$$\frac{1}{\tau}=2\pi\rho(E_F)M\ ,$$

$\rho(E_F)$ being the density of states at the Fermi surface in the $N=\infty$ limit. In Eq. (2.5), μ, the chemical potential, has already absorbed an infinite shift.

Using this Green's function, *which we denote hereafter with a thin line,* we can derive a Dyson equation for the one-particle Green's function $g_\infty(\vec{\mathrm{p}},\omega)$ in the interacting case at $N=\infty$. The Dyson equation is

$$g_\infty^{-1}(\vec{\mathrm{p}},\omega)=G_\infty^{-1}(\vec{\mathrm{p}},\omega)-\Sigma_\infty(\vec{\mathrm{p}},\omega)\ , \tag{2.6}$$

where the contribution to self-energy $\Sigma_\infty(\vec{\mathrm{p}},\omega)$ is shown in Fig. 5. This self-energy is a real constant and so its effect is just to shift the chemical potential.

A priori there are other contributions to the self-energy like the graph shown in Fig. 6. However, their contribution is zero if the standard procedure[9,11] of approximating momentum integrations by energy integrations times the density of states at the Fermi surface $\rho_\infty(E_F)$ is used. For example, the graph in Fig. 6 implies

$$\delta\Sigma=\int_q G_\infty^+(\vec{\mathrm{p}}-\vec{\mathrm{q}},\omega)G_\infty^+(\vec{\mathrm{p}}-\vec{\mathrm{q}},\omega)MgnU(0)\ , \tag{2.7}$$

where G^+ is the Green's function above the cut ($\omega>0$) and n is the average particle density. The standard procedure approximates this as

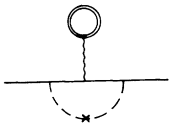

FIG. 6. A contribution to the real part of the self-energy.

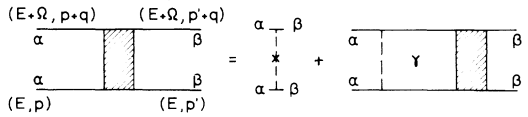

FIG. 7. Integral equation (2.9).

$$\delta\Sigma\approx MgnU(0)\rho_\infty(E_F)\int_{-\infty}^{\infty}d\epsilon\,[G_\infty^+(\epsilon,\omega)]^2\ . \tag{2.8}$$

The integral is equal to zero since both poles are on the same side of the cut in the complex plane. If this procedure is not adopted, then Fig. 6 does give a real contribution. But this can again be absorbed by a shift in the chemical potential. Thus by redefining the chemical potential, the total one-particle Green's function g_∞ of the interacting system is equal to that of the noninteracting system G_∞, Eq. (2.5).

2. Two-particle Green's functions: the diffusive mode $K_{\alpha\beta\beta\alpha}$ (Ref. 6)

At $N=\infty$ the only contributions to the diffusive modes [Fig. 2(a)] are those of the noninteracting theory.[6] The two-particle Green's function $K^\infty_{\alpha\beta\beta\alpha}$ obeys the equation (Fig. 7)

$$K=\frac{M}{N}+M\left[\int_{\vec{\mathrm{k}}}G_\infty(\vec{\mathrm{k}}+\vec{\mathrm{p}},\omega+\Omega)G_\infty(\vec{\mathrm{k}},\Omega)\right]K\ , \tag{2.9}$$

where we have dropped the indices. Thus at $N=\infty$ it reduces to the ladder approximation.[9] The solution to Eq. (2.9) is

$$K^{\pm\pm}_{\alpha\beta\beta\alpha}(\rho,\Omega)=\frac{M}{N}\ , \tag{2.10a}$$

$$K^{\pm\mp}_{\alpha\beta\beta\alpha}(\vec{\mathrm{p}},\Omega)=\frac{M}{N\tau}\frac{1}{D\vec{\mathrm{p}}^{\,2}-i\,|\,\Omega\,|}\ , \tag{2.10b}$$

where $K^{\pm\pm}$ indicates that both electronic lines are on the same side of the cut and $K^{\pm\mp}$ indicates that they are on opposite sides of the cut. In Eq. (2.10b) D is the diffusion constant, which to this order is equal to

$$D=\frac{v_F^2\tau}{d} \tag{2.11}$$

(v_F is the Fermi velocity). The pole at small $\vec{\mathrm{p}}$ and Ω in (2.10b) reflects the diffusive behavior of the system at $N=\infty$. The conductivity is related to the diffusion constant by the Einstein relation

$$\sigma=\frac{\partial n}{\partial\mu}D \tag{2.12}$$

(we have set $e^2/\hbar=1$). In this system $\partial n/\partial\mu=\rho_\infty(E_F)$. Thus there is no localization (near E_F) at $N=\infty$.[6]

3. The vertex function Γ

The interaction vertex Γ obeys the Dyson equation (Fig. 8),

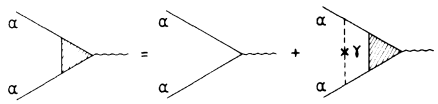

FIG. 8. Vertex function Γ at $N=\infty$.

$$\Gamma^{\alpha\alpha}=1+M\left[\int_{\vec{k}} G_\infty(\vec{k}+\vec{q},\omega+\Omega)G_\infty(\vec{k},\Omega)\right]\Gamma^{\alpha\alpha} . \tag{2.13}$$

The solution to (2.13) is

$$\Gamma^{\alpha\alpha}_{\pm\pm}=1 , \tag{2.14a}$$

$$\Gamma^{\alpha\alpha}_{\pm\mp}=\frac{1}{\tau(Dq^2-i\,|\,\Omega\,|\,)} . \tag{2.14b}$$

4. *Effective interaction*

The effective interaction potential at $N=\infty$, $U_\infty(\vec{p},\omega)$, is found to satisfy the RPA. The Dyson equation is (Fig. 9)

$$U_\infty(\vec{p},\infty)=U(\vec{p},\omega)+gU(\vec{p},\omega)\Pi_\infty(\vec{p},\omega)U_\infty(\vec{p},\omega) , \tag{2.15}$$

where the polarization bubble is

$$\Pi_\infty(\vec{p},\omega)=-i\int\frac{d\Omega}{2\pi}\int\frac{d^dq}{(2\pi)^d}G_\infty(\vec{p}+\vec{q},\omega+\Omega)\times G_\infty(\vec{q},\Omega)\Gamma(\vec{p},\omega+\Omega,\Omega) . \tag{2.16}$$

The result is

$$\Pi_\infty(\vec{p},\omega)=-\rho_\infty(E_F)\frac{D\vec{p}^{\,2}}{D\vec{p}^{\,2}-i\,|\,\omega\,|} . \tag{2.17}$$

The effective interaction at $N=\infty$ is then

$$U_\infty(\vec{p},\omega)=\frac{U(\vec{p},\omega)}{1-g\Pi_\infty(\vec{p},\omega)U(\vec{p},\omega)} . \tag{2.18}$$

For $\vec{p},\omega$ small we find

$$U_\infty(p,\omega)\approx\frac{1}{\lambda^2}\frac{D\vec{p}^{\,2}-i\,|\,\omega\,|}{D\vec{p}^{\,2}-i(\lambda_0^2/\lambda^2)\,|\,\omega\,|} , \tag{2.19}$$

with λ^2 equal to

$$\lambda^2=\lambda_0^2+g\rho_\infty(E_F) . \tag{2.20}$$

In the limit $\vec{p}\to 0$ (ω finite) we recover the unperturbed result $U_\infty(\vec{p},\omega)\sim\lambda_0^{-2}$. In the opposite limit ($\omega\to 0$, $\vec{p}$ finite) we get $U_\infty(\vec{p},\omega)\sim\lambda^{-2}$.

In fact, this relation between the two limits (i.e., $\lambda^2=\lambda_0^2+g_\infty C_\infty$) is expected to hold even for finite N. To see this, we note that it is really $-\Pi(\vec{p}\to 0,\omega=0)$ that enters in Eq. (2.20). From linear response,

$$\delta n(\vec{p},\omega)=\tilde{\Pi}(\vec{p},\omega)\delta\phi(\vec{p},\omega) , \tag{2.21}$$

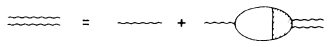

FIG. 9. Effective interaction at $N=\infty$.

where δn is the impurity-averaged change in density due to a potential disturbance $\delta\phi$. The retarded density correlation function $\tilde{\Pi}$ is related to the time-ordered polarization Π by[12]

$$\mathrm{Re}\tilde{\Pi}=\mathrm{Re}\Pi ,$$

$$\mathrm{Im}\tilde{\Pi}=(\mathrm{sgn}\omega)\mathrm{Im}\Pi . \tag{2.22}$$

For a static potential $\delta\phi$ is equivalent to a change in the chemical potential $\delta\mu=-\delta\phi$, hence[13]

$$\tilde{\Pi}(p\to 0,0)=-\frac{\partial n}{\partial\mu} . \tag{2.23}$$

For finite N, $\partial n/\partial\mu$ no longer equals ρ_∞ but the difference is finite and nonsingular. In the next section we will see that indeed the $1/N$ correction to $\Pi(\vec{p}\to 0,0)$ is nondivergent. Nevertheless, there will be important corrections to $U(\vec{p},\omega)$ to order $1/N$. Let us expand U_∞ in the limit $Dp^2\ll|\,\omega\,|$; we have

$$U_\infty(\vec{p},\omega)\approx\frac{1}{\lambda_0^2}\left[1+\frac{Dp^2}{i\,|\,\omega\,|}(\lambda^2/\lambda_0^2-1)\right] . \tag{2.24}$$

The coefficient of $Dp^2/i\,|\,\omega\,|$, which will henceforth be called ν, is equal to λ^2/λ_0^2-1, at $N=\infty$. This will no longer be true for finite N.

Thus at $N=\infty$ we find a weakly disordered metal. The quasiparticles have a finite lifetime τ, the two-particle Green's functions exhibit diffusive behavior, and the system conducts. The electrons interact via the bare g/λ_0^2 potential in the dynamic limit ($p\to 0$) and via the "screened" g/λ^2 potential in the static limit ($\omega\to 0$). There is no transition at $N=\infty$ between metallic and insulating behavior, a result already found by Wegner[6] in the noninteracting theory.

We will use this $N=\infty$ limit as the starting point for an expansion around a metallic phase. Technically, this will be done in the following sections by means of a $1/N$ expansion. It should be noted that this $1/N$ expansion, as in the noninteracting case, is of a very different nature from the $1/N$ expansion in $0(N)$ symmetric ferromagnets. In that case there is a transition at $N=\infty$ and its character changes smoothly, in a calculable way, for large N. This is not the case here. The theory is finite at $N=\infty$ and divergencies appear at finite N. Thus a renormalization procedure will be needed. As a matter of fact, the $1/N$ expansion for the model we are studying is close in spirit to the loop expansion around the broken-symmetry phase in ferromagnets.[14]

III. THE $1/N$ EXPANSION

In the preceding section we discussed the properties of the model in the limit $N\to\infty$. It was shown that so far as

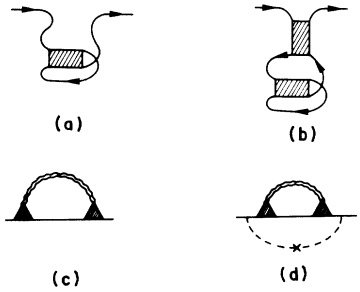

FIG. 10. $1/N$ contributions to the one-particle Green's function.

$N=\infty$ the system is just a weakly disordered metal. In this section we perform a $1/N$ expansion for the Green's functions. We are going to see that the $1/N$ corrections alter in a rather profound way the physics of this problem. Since in the $N=\infty$ limit the system is a weakly disordered metal, the $1/N$ expansion is a natural tool to investigate the stability of this state.

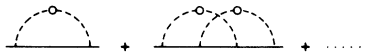

FIG. 11. Another way of drawing Fig. 10(a).

A. One-particle Green's function

In Fig. 10 we present all the contributions to the one-particle Green's function to order $1/N$. The shaded boxes and triangles of Fig. 10 represent the $N=\infty$ two-particle Green's function of Fig. 7 and the $N=\infty$ vertex of Fig. 8, respectively. The wavy lines are the full $N=\infty$ effective interaction U_∞. A more direct way of drawing Fig. 10(a) is shown in Fig. 11 [there is an analogous representation of Fig. 10(b)].

From the graphs shown in Fig. 10 only 10(c) and 10(d) are infrared divergent in two dimensions. Hence the contributions of 10(a) and 10(b) will be dropped.

Diagram 10(c) has a contribution equal to, for $\epsilon>0$,

$$\delta\Sigma_1=\frac{ig}{N}[G_\infty(\vec{p},\epsilon)]\int\frac{d^2k}{(2\pi)^2}\int\frac{d\Omega}{2\pi}[\Gamma_\infty(\vec{k},\Omega)]^2G_\infty(\vec{p}-\vec{k},\epsilon-\Omega)U_\infty(\vec{k},\Omega)\,, \tag{3.1}$$

where $\Gamma_\infty(\vec{k},\Omega)$ is given in Eqs. (2.14a) ($\epsilon-\Omega>0$) and (2.14b) ($\epsilon-\Omega<0$). This graph has been calculated by Altshuler *et al.*[4] who found a logarithmic infrared divergence when $\epsilon-\Omega<0$. By keeping *all divergent contributions,* one finds in Fig. 10(c)

$$\delta\Sigma_1\cong\frac{g}{N\lambda^2\tau^2}G_-^\infty(\vec{p},\epsilon)\frac{1}{8\pi^2D}\frac{\ln(\lambda^2/\lambda_0^2)}{\lambda_0^2/\lambda^2-1}\left[-\ln\frac{|\epsilon|}{E_c}\right]$$
$$+\frac{ig}{N\lambda^2\tau^2}\frac{E_c}{8\pi^2D}\frac{[G_-^\infty(\vec{p},\epsilon)]^2}{2mD}\left[1+\frac{\vec{p}^{\,2}}{m}G_-^\infty(\vec{p},\epsilon)\right]\left[1+\frac{\epsilon}{E_c}\left[\ln\frac{\epsilon}{E_c}-1\right]\right]+\cdots\,, \tag{3.2}$$

where the ellipsis indicates finite terms and where we have introduced a bandwidth cutoff E_c. In (3.2) G_+^∞ (G_-^∞) is the $N=\infty$, retarded (advanced) one-particle Green's function.

Graph 10(d) gives a contribution to the one-particle Green's function equal to ($\epsilon>0$)

$$\delta\Sigma_2=M\int\frac{d^2p}{(2\pi)^2}[G_+^\infty(\vec{p},\epsilon)]^2\delta\Sigma_1\,. \tag{3.3}$$

An explicit calculation yields [Fig. 10(d)]

$$\delta\Sigma_2\cong\frac{i}{8\pi^2D}\frac{g}{N\lambda^2\tau}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda_0^2/\lambda^2)-1}\ln\left[\frac{\epsilon}{E_c}\right]-\frac{3g\rho}{N\lambda^2}\frac{1}{8\pi^2\rho D}\epsilon\ln\frac{\epsilon}{E_c}\,. \tag{3.4}$$

We can summarize results (3.2) and (3.4) by writing the bare one-particle irreducible two-point function $G_B^{-1}(\vec{p},\epsilon)$ as

$$G_B^{-1}(\vec{p},\epsilon)=G_\infty^{-1}(\vec{p},\epsilon)-\Sigma(\vec{p},\epsilon)\,. \tag{3.5}$$

The result is

$$G_B^{-1}(\vec{p},\epsilon)=\epsilon\left[1-\frac{1}{\pi N}\frac{1}{2\pi\rho D}\left[\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}+\frac{5}{4}\frac{\rho g_{+-}}{\lambda^2}\right]\ln\frac{\epsilon}{E_c}\right]$$
$$-[E(\vec{p})-\mu]\left[1-\frac{1}{\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}\ln\frac{\epsilon}{E_c}\right]$$
$$+\frac{i}{2\tau}\left[1-\frac{1}{\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}\ln\frac{\epsilon}{E_c}\right], \tag{3.6}$$

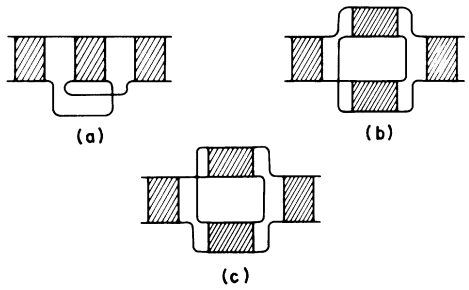

FIG. 12. $1/N$ noninteracting contributions to K_{+-}.

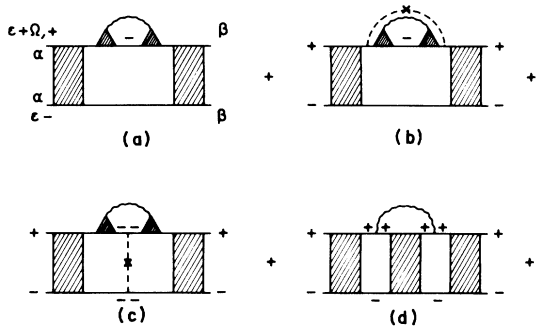

FIG. 13. Interaction contributions to the diffusive mode. Four extra graphs are obtained by exchanging the top and bottom lines.

where we have set $\rho \equiv \rho_\infty(E_F)$ and we have introduced a coupling constant g_{+-} for the processes between electrons with positive and negative frequencies. At this level $g_{++}=g_{+-}=g$. But as we are going to show below, these two coupling constants *renormalize differently*. We are only anticipating this result.

A glance at Eq. (3.6) reveals the presence of an infrared-divergent renormalization of the mean free time τ. In fact, Eq. (3.6) suggests that, at the fixed point, τ should go to zero. This result is new and is a consequence of diagram 10(d). Notice that this diagram could have been dropped in a naive expansion (it contains "more disorder") even though it has the *same* degree of infrared divergence as diagram 10(c). It is an important property of the $1/N$ expansion that all diagrams with the *same degree of infrared divergence appear in the same order in* $1/N$. This property holds for all the operators that we have studied to low orders in $1/N$. We conjecture that this may be a property of this expansion to all orders in $1/N$. In this sense the $1/N$ expansion is consistent.

The vanishing of τ, and hence of the mean free path, implies that the two-point function decays, in real space, over a distance comparable with the cutoff. Thus the memory of the phase of the one-body wave functions is lost on a distance of the order of the lattice spacing. Wegner, in his original paper on the N-orbital model,[6] introduced a model which had this property built in as a consequence of a local gauge invariance. Interaction effects force this property on us instead. The vanishing of the mean free path can be understood on physical grounds if we recall that each impurity, even if it is pointlike at the bare level, acquires an "electronic cloud" arising from correlation effects. The average distance a dressed electron has to travel between two impurity scatterings is, hence, reduced. Equation (3.6) says that this distance has to go to zero at the fixed point.

B. Two-particle Green's function $K^{+-}_{\alpha\beta\beta\alpha}$

The Feynman diagrams that contribute, to order $1/N$, to the two-particle Green's function $K^{+-}_{\alpha\beta\beta\alpha}$ (i.e., the diffusive mode) are depicted in Fig. 12.

Diagrams 12(a)–12(c) represent the $1/N$ contribution of the noninteracting sector of the theory and correspond to the insertion of maximally crossed ladders, in the language of Abrahams *et al.*[2] Within the present context these diagrams were considered by Oppermann and Wegner.[7] Their contribution to $K^{+-}_{\alpha\beta\beta\alpha}$ is

$$\delta K_1^{+-} = \frac{1}{N}\frac{M}{N\tau(D\vec{q}^{\,2}-i\Omega)}\frac{1}{Dq^2-i\Omega} \times\left[-\frac{1}{2\pi}\frac{1}{2\pi\rho D}D\vec{q}^{\,2}\ln\frac{|\Omega|}{D\Lambda^2}\right], \tag{3.7}$$

where we have introduced a high momentum transfer cutoff Λ. Dimensionally we can use either $D\Lambda^2$ or the bandwidth cutoff E_c. Both schemes differ only by finite contributions.

Next we have to consider the effects due to both interactions and disorder. They are presented in the diagrams of Fig. 13. Their total contribution is

$$\delta K_2^{+-} = \left[\frac{1}{N}\frac{M}{N\tau(D\vec{q}^{\,2}-i\Omega)}\frac{1}{Dq^2-i\Omega}\right]X, \tag{3.8}$$

where X is given by

$$\begin{aligned}X=&\frac{1}{4\pi}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}(Dq^2-i\Omega)\left[\ln\frac{|\epsilon|}{E_c}+\ln\frac{|\epsilon+\Omega|}{E_c}\right]\\&+\frac{i}{4\pi}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda^2}\left[(\epsilon+\Omega)\ln\left|\frac{\epsilon+\Omega}{E_c}\right|-\epsilon\ln\left|\frac{\epsilon}{E_c}\right|\right]+\frac{1}{2\pi}\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}(-Dq^2-i\Omega)\ln\frac{|\Omega|}{D\Lambda^2}\\&-\frac{i}{4\pi}\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda^2}\left[(\epsilon+\Omega)\ln\left|\frac{\epsilon+\Omega}{E_c}\right|-\epsilon\ln\frac{|\epsilon|}{E_c}-2\Omega\ln\frac{|\Omega|}{D\Lambda^2}\right]\\&-\frac{1}{\pi}\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda_0^2}\left[\frac{1}{(\lambda^2/\lambda_0^2)-1}-\frac{\lambda^2}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2-1)^2}\right]dq^2\ln\frac{|\Omega|}{D\Lambda^2}.\end{aligned} \tag{3.9}$$

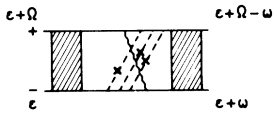

FIG. 14. A contribution which does not conserve the energy of each line separately.

The last three terms in (3.9) have been computed in the limit $Dq^2 \ll \Omega$, the dynamic regime.

The contribution of Eq. (3.7) is of course the same one that lead Abrahams *et al.*[2] to their β function. It has been argued by a number of authors, notably by Wegner,[15] Khmelnitzkii,[16] Houghton *et al.*,[17] McKane and Stone,[18] and Hikami,[19] that, in the noninteracting case, it is only necessary to renormalize the diffusion constant and that there is no wave-function renormalization of the diffusive mode K^{+-}. The form of Eq. (3.9) clearly shows that this result is not valid when interactions are present.

All the graphs shown in Figs. 12 and 13 have the property that the energy of each electronic line is separately conserved. This is a property that cannot be maintained in an interacting theory. In fact there are contributions to K^{+-} that violate this property. One such graph is shown in Fig. 14. These graphs change the nature of the scattering process. To order $1/N$ they can be ignored in K^{+-}. To next order in $1/N$ one would have to worry about renormalizing these new terms. We will not consider this problem here. It is important to note, however, that even though graph 14 is not important in K^{+-} (to order $1/N$) it is important in the vertex functions.

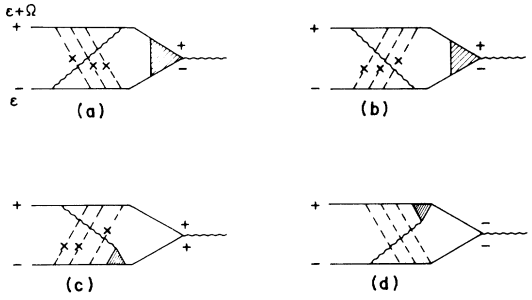

FIG. 15. $1/N$ corrections to the Γ^{+-} vertex.

C. Vertex corrections

At $N=\infty$ we showed that two vertex functions, namely Γ^{++} and Γ^{+-}, appear naturally. The coupling was, however, identical for both processes. This property of the $N=\infty$ theory is already lost at the level of the $1/N$ corrections.

1. The Γ^{+-} vertex

The (infrared-divergent) $1/N$ corrections to the Γ^{+-} vertex are shown in Fig. 15. The bare Γ^{+-} vertex is, omitting the diffusive factor,

$$\Gamma^{+-}=\frac{\sqrt{\rho g_{+-}}}{\lambda^2}\left[1+\left[-\frac{1}{\pi N}\right]\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda^2}\ln\frac{|\Omega|}{D\Lambda^2}\right.$$
$$\left.+\frac{-1}{2\pi N}\frac{1}{2\pi\rho D}\frac{\rho}{\lambda^2}\sqrt{g_{++}g_{+-}}\left\{\ln\left[\Omega+(\epsilon+\Omega)\left[1-\frac{\lambda_0^2}{\lambda^2}\right]\right]+\ln\left[\Omega-\epsilon\left[1-\frac{\lambda_0^2}{\lambda^2}\right]\right]\right\}\right], \tag{3.10}$$

where $\epsilon+\Omega>0$ and $\epsilon<0$. The diagrams 15(a)–15(d) have been evaluated in the limit of zero momentum transfer. Notice that when writing (3.10) we have assumed a normalization of Γ^{+-} which *included* the coupling constant g_{+-}. The diffusive factor, omitted in (3.10), renormalizes like the diffusive mode K^{+-}.

There are, of course, other diagrams which correspond to building up the diffusive mode. The graphs we consider are "one-particle irreducible" in the diffusive mode. In a physical sense Γ^{+-} represents the coupling between the diffusive mode and the "plasma oscillation" represented by the wavy line.

2. The Γ^{++} vertex

The $1/N$ corrections to the Γ^{++} vertex can be computed in a similar fashion. The divergent contributions are shown in Fig. 16. The total bare Γ^{++} vertex, to order $1/N$, is

$$\Gamma^{++}=\frac{\sqrt{\rho g_{++}}}{\lambda^2}\left[1-\left[\frac{2}{\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{\lambda^2/\lambda_0^2-1}+\frac{3}{2\pi N}\frac{1}{2\pi\rho D}\frac{\rho}{\lambda^2}\sqrt{g_{++}g_{+-}}\right]\ln\frac{\Omega}{E_c}\right]+\frac{\rho g_{++}}{\lambda^2}\frac{A}{i\Omega\tau}, \tag{3.11}$$

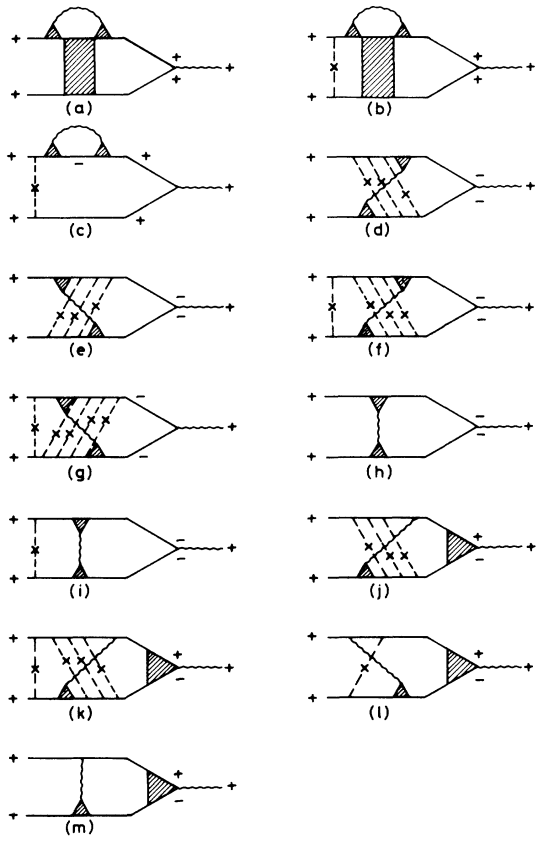

FIG. 16. $1/N$ corrections to the Γ^{++} vertex. There are three other graphs which are obtained by exchanging the top and bottom lines in (a), (b), and (c).

where

$$A=\frac{1}{4\pi N}\frac{\rho g_{+-}}{\lambda_0^2}\frac{1}{2\pi\rho D}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}\ln\left|\frac{\epsilon+\Omega}{\epsilon}\right| . \tag{3.12}$$

Equation (3.11) is quite important. It says that in addition to a coupling-constant renormalization, implied by the brackets, the $1/N$ corrections have changed the nature of the vertex: $1/\Omega$ term has been generated. The coefficient A is finite. In the small-Ω limit this term becomes dominant. Equation (3.11) in fact implies that the particle-particle channel acquires a diffusive component. This result, which is a fluctuation effect, should be contrasted with the one-particle theory in which the particle-particle channel is always nondiffusive. This is closely connected with the smoothness of the one-particle Green's function in the noninteracting theory.

This, however, is not the end of the story. For the theory to be renormalizable we must have a *finite* number of relevant operators. Thus it becomes necessary to know if more operators do not arise in higher orders in the $1/N$ expansion.

We cannot answer this question within the framework of this one-loop (i.e., $1/N$) calculation. However, if one *assumes* that the singularities in the electron self-energy are of the form $(\Omega/E_c)^p\ln^q(\Omega/E_c)$ and if one recalls the Ward identity,[20,21]

$$k_\mu\Gamma_\mu\left[\vec{p}+\frac{k}{2},\omega+\frac{\Omega}{2};\vec{p}-\frac{k}{2},\omega-\frac{\Omega}{2}\right] =G^{-1}\left[\vec{p}+\frac{k}{2},\omega+\frac{\Omega}{2}\right]-G^{-1}\left[\vec{p}-\frac{k}{2},\omega-\frac{\Omega}{2}\right], \tag{3.13}$$

(using the notation of Ref. 19), then one can readily conclude that the vertex function Γ^{++} will always be of the form $C+A/i\Omega$ (in the zero momentum transfer limit) with logarithmic renormalizations of the constants C and A. Of course it remains to be shown that the electron self-energy does have that singularity structure. A calculation to higher orders in $1/N$ becomes necessary. We will not elaborate more about this here.

D. $1/N$ corrections to the effective potential

There are divergent contributions to the polarization operator Π, and hence to the potential, to order $1/N$. The relevant diagrams are obtained by closing the external legs in the two-particle Green's function.

In the *static limit* ($Dq^2\gg\Omega$) the corrections to Π_S are

$$\delta\Pi\approx-\rho\frac{i\Omega}{Dq^2}\left[-\frac{1}{2\pi N}\frac{1}{2\pi\rho D}\ln\frac{\Omega}{E_c}\right], \tag{3.14}$$

where $\rho=\rho_\infty(E_F)$.

In the *dynamic limit* ($Dq^2\ll\Omega$) the corrections have the form

$$\delta\pi_D\approx-\rho\frac{i}{Dq^2}\left[\frac{1}{2\pi N}\frac{1}{2\pi\rho D}\ln\frac{\Omega}{E_c}+\frac{1}{2\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}\ln\frac{\Omega}{E_c}+\frac{3}{2\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda_0^2}\frac{\ln(\lambda^2/\lambda_0^2)}{(\lambda^2/\lambda_0^2)-1}\ln\frac{\Omega}{E_c}$$
$$-\frac{1}{\pi N}\frac{1}{2\pi\rho D}\frac{\rho}{\lambda_0^2}\sqrt{g_{++}g_{+-}}\frac{\ln(\lambda^2/\lambda_0^2)}{\lambda^2/\lambda_0^2-1}\ln\frac{\Omega}{E_c}-\frac{1}{2\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{+-}}{\lambda^2}\ln\frac{\Omega}{E_c}+\frac{1}{2\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda^2}\ln\frac{\Omega}{E_c}$$
$$+\frac{1}{\pi N}\frac{1}{2\pi\rho D}\frac{\rho g_{++}}{\lambda_0^2}\frac{1-\dfrac{\ln(\lambda^2/\lambda_0^2)}{1-\lambda_0^2/\lambda^2}}{\lambda^2/\lambda_0^2-1}\ln\frac{\Omega}{E_c}\Bigg] . \tag{3.15}$$

Please note that $\delta\Pi \to 0$ both if $\Omega \to 0$ (q fixed) or $q \to 0$ (Ω fixed). This result is a consequence of particle-number conservation.

IV. RENORMALIZATION

In the preceding section we presented all the infrared-divergent contributions that, in two dimensions, appear in all physically relevant operators. In this section we will construct a renormalization scheme that, to order $1/N$, will control the divergencies in all these operators.

In principle, in order to renormalize the theory we have to define the coupling constants, Green's functions, etc., for some value of momenta and frequency, i.e., we have to choose the renormalization point. We choose to renormalize the theory in the dynamic regime ($Dq^2 \ll |\omega|$) where there is no screening. We are going to show below that this procedure is consistent since a "screening parameter" ν can be introduced and it is irrelevant at the critical fixed point.

In the noninteracting theories it has been shown[16–19] that it is only necessary to renormalize the diffusion constant (or rather a quantity proportional to its inverse) to render the theory finite. However, the results of Sec. III clearly show that this is not enough when interactions are present.

Two more coupling constants need to be renormalized as well as the mean free time τ, the Fermi velocity v_F, and the screening parameter ν. Wave-function renormalizations of the one-particle Green's function and of the diffusive mode are also necessary. To the leading order in $1/N$ this is all we need. It remains to be proved that this procedure is consistent in higher orders in $1/N$. We will return to this point below.

A. Renormalization of the fermion propagator

The one-particle Green's function requires a wave-function renormalization Z_ψ of the Fermi field ψ, together with a renormalization of the Fermi velocity v_F and of the mean free time τ. Specifically, we *define* the renormalized, and hence finite, electron propagator to be

$$G_R^{-1}(\vec{p},\omega)=\omega-E_R(\vec{p})+\mu_R+\frac{i}{2\tau_R} . \tag{4.1}$$

Introducing the wave-function renormalization Z_ψ and the renormalization constant Z_τ and Z_{v_F} for the mean free time and Fermi velocity, respectively, we can write

$$G_R^{-1}(\vec{p},\omega)=Z_\psi G_B^{-1}(\vec{p},\omega) . \tag{4.2}$$

Equations (4.1) and (4.2) imply the renormalization conditions

$$\mathrm{Im}G_R^{-1}\Big|_{\omega=0,\ |\vec{p}|=p_F}=\frac{1}{2\tau_R} , \tag{4.3a}$$

$$\frac{\partial}{\partial\omega}\mathrm{Re}G_R^{-1}\Big|_{\omega=0,\ |\vec{p}|=p_F}=1 , \tag{4.3b}$$

$$-\frac{\partial}{\partial|\vec{p}|}\mathrm{Re}G_R^{-1}\Big|_{\omega=0,\ |\vec{p}|=p_F}=v_F^R . \tag{4.3c}$$

The renormalized and bare values of the mean free time and Fermi velocity are related through the renormalization constants,

$$\tau_R=\tau Z_\tau Z_\psi^{-1} , \tag{4.4a}$$

$$v_F^R=v_F Z_{v_F} Z_\psi . \tag{4.4b}$$

The renormalization of the mean free time is a consequence of diagram 10(d).

By making explicit use of Eq. (3.6) one finds the following expressions:

$$Z_\psi=1+\frac{1}{\pi N}xz\left[\frac{\ln(1+\nu)}{\nu}+\frac{5}{4}\frac{1}{1+\nu}\right]\ln\left|\frac{\omega}{E_c}\right| , \tag{4.5a}$$

$$Z_\tau=1+\frac{1}{2\pi N}xz\frac{\ln(1+\nu)}{\nu}\ln\left|\frac{\omega}{E_c}\right| , \tag{4.5b}$$

$$Z_{v_F}=1-\frac{1}{\pi N}xz\frac{\ln(1+\nu)}{\nu}\ln\left|\frac{\omega}{E_c}\right| , \tag{4.5c}$$

where we have introduced the (bare) parameters,

$$x=\frac{1}{2\pi\rho D} , \tag{4.6a}$$

$$z=\frac{\rho g_{+-}}{\lambda_0^2} , \tag{4.6b}$$

$$y=\frac{\rho g_{++}}{\lambda_0^2} , \tag{4.6c}$$

and

$$\nu=\frac{\lambda^2}{\lambda_0^2}-1 . \tag{4.6d}$$

Dimensionally, x scales like L^{d-2} (or $\omega^{(2-d)/2}$). Similarly one can see that y and z *are dimensionless in all dimensions.*

B. Renormalization of the diffusive mode K^{+-}

It was mentioned at the beginning of this section that in the noninteracting theory a renormalization of the diffusion constant is enough to render the theory finite. In fact it has been shown by the same authors,[13,16–19] that in the noninteracting theory there is no wave-function renormalization of the diffusive mode K^{+-}. We have argued in Sec. III B that this is no longer true when interactions are taken into account.

Define a renormalized diffusive mode K_R^{+-},

$$K_R^{+-}=\frac{M}{N\tau_R}\frac{1}{D_R q^2-i\Omega} . \tag{4.7}$$

Equation (4.7) is equivalent to the assumption that the metallic phase is stable. Equation (4.7) implies the set of renormalization conditions,

$$\frac{M}{\tau_R}\frac{\partial}{\partial\Omega}(K_R^{+-})^{-1}=-i\ , \tag{4.8a}$$

$$\frac{M}{\tau_R}\frac{\partial}{\partial q^2}(K_R^{+-})^{-1}=D_R\ . \tag{4.8b}$$

This parameter D_R should not be confused, however, with the renormalized diffusion constant even though they have the same dimension. The renormalized diffusion constant, which is linked with the renormalized conductivity via the Einstein relation, is extracted from a response function. When doing so one finds that there are contributions not only from momenta and frequencies such that $\omega/(Dq^2)\gg 1$ but from other regimes too. These contributions are sufficient to make our diffusion parameter D_R different from the diffusion constant. The same comment applies for the dimensionless "resistance" x defined below.

Two renormalization constants, Z_D and Z_Q, are necessary to make K^{+-} finite:

$$K_R^{+-}=Z_\psi^2 Z_Q^{-1}K_B^{+-}\ , \tag{4.9a}$$

$$x_R=xZ_\psi Z_\tau Z_Q^{-1}Z_D^{-1}E_c^{-\epsilon/2}\ , \tag{4.9b}$$

where x is the dimensionless "resistance" and $\epsilon=d-2$. The renormalization constants, to leading order in $1/N$, are given by

$$Z_Q=1+\left\{\frac{2}{\pi N}xz\frac{\ln(1+\nu)}{\nu}+\frac{1}{\pi N}\frac{xz}{1+\nu}+\frac{xy}{2\pi N}\frac{\ln(1+\nu)}{\nu}-\frac{1}{4\pi N}\frac{xy}{1+\nu}\right\}\ln\left|\frac{\Omega}{E_c}\right|\ , \tag{4.10a}$$

$$Z_D=1+\left\{\frac{x}{2\pi N}+\frac{xy}{2\pi N}\frac{\ln(1+\nu)}{\nu}+\frac{xy}{\pi N\nu}\left[1-\left[\frac{1+\nu}{\nu}\right]\ln(1+\nu)\right]-\frac{xz}{2\pi N}\frac{\ln(1+\nu)}{\nu}\right\}\ln\left|\frac{\Omega}{E_c}\right|\ . \tag{4.10b}$$

C. Renormalization of the vertices Γ^{++} and Γ^{+-}

The vertices Γ^{++} and Γ^{+-} can be made finite, to leading order in $1/N$, by means of a multiplicative renormalization of the coupling constants $(\rho g_{++})/\lambda_0^2$ and $(\rho g_{+-})/\lambda_0^2$ as follows:

$$y_R=yZ_\psi^2 Z_y^{-1}\ , \tag{4.11a}$$

$$z_R=zZ_z^{-1}Z_\psi^{-2}\ . \tag{4.11b}$$

The difference in the renormalization prescription for y and z originates in the different physical character of both coupling constants. While g_{++} measures the strength of particle-particle (or hole-hole) scattering, the coupling constant g_{+-} measures the strength of the coupling between the diffusive mode K_{+-} and the density fluctuations.

Z_y and Z_z are equal to

$$Z_y=1+\left\{\frac{4}{\pi N}xz\frac{\ln(1+\nu)}{\nu}+\frac{3}{\pi N}\frac{x\sqrt{yz}}{1+\nu}\right\}\ln\left|\frac{\Omega}{E_c}\right|\ , \tag{4.12a}$$

$$Z_z=1+\frac{2}{\pi N}\left\{\frac{xy}{1+\nu}+x\sqrt{yz}\frac{\ln(1+\nu)}{\nu}\right\}\ln\left|\frac{\Omega}{E_c}\right|\ . \tag{4.12b}$$

D. Renormalization of the effective interaction

The divergencies found in Π and hence in U (cf. Sec. III D) can be absorbed by means of a multiplicative renormalization of the (dimensionless) "screening" parameter ν. In Sec. II we have shown that a multiplicative renormalization of the polarization operator Π, as suggested by McMillan,[5] is not possible since it is forbidden by conservation laws.

We choose to renormalize the theory in the dynamic regime ($Dq^2\ll\Omega$). The renormalized and bare values of ν are related by

$$\nu_R=\nu Z_\nu^{-1}\frac{D}{D_R}\ , \tag{4.13}$$

where Z_ν is found to be, to leading order in $1/N$,

$$Z_\nu=1-\left\{\frac{x}{2\pi N}+\frac{1}{2\pi N}xz\frac{\ln(1+\nu)}{\nu}+\frac{3}{2\pi N}xy\frac{\ln(1+\nu)}{\nu}-\frac{1}{\pi N}x\sqrt{yz}\frac{\ln(1+\nu)}{\nu}-\frac{1}{2\pi N}\frac{xz}{1+\nu}+\frac{1}{2\pi N}\frac{xy}{1+\nu}+\frac{1}{\pi N}\frac{xy}{\nu}\left[1-\left[\frac{\nu+1}{\nu}\right]\ln(1+\nu)\right]\right\}\ln\left|\frac{\Omega}{E_c}\right|\ . \tag{4.14}$$

V. RENORMALIZATION-GROUP EQUATIONS AND SCALING

The renormalization procedure developed in Sec. IV assumes the stability of the metallic phase. In order to check the validity of this assumption one must resort to the renormalization group.

The renormalization-group procedure that we use relates different bare theories at different values of the cutoff keeping the renormalized theory fixed. This is a standard procedure that has been considerably discussed in the literature[14] and we shall not elaborate on it.

A. Renormalization-group equations

From Eqs. (4.9b), (4.11a), (4.11b), and (4.13) we can extract the β functions,

$$\beta_x = -E_c \left. \frac{\partial x}{\partial E_c} \right|_R , \tag{5.1}$$

with similar definitions for β_y, β_z, and β_ν. The subscript R means that renormalized quantities are kept fixed as the cutoff E_c is changed. Please note that by dimensional analysis it is always possible to change from a scheme with a bandwidth cutoff E_c to another one with momentum transfer cutoff Λ.

The β functions are equal to

$$\beta_x = -\frac{\epsilon x}{2} + \frac{x^2 y}{2\pi N} + \frac{x^2}{\pi N}\left\{\frac{1}{\nu}\left[1-\left[\frac{\nu+1}{\nu}\right]\ln(\omega+1)\right] + \frac{\ln(1+\nu)}{\nu} - \frac{1}{4(1+\nu)}\right\} - \frac{x^2 z}{4\pi N}\frac{1}{(1+\nu)} , \tag{5.2a}$$

$$\beta_y = \frac{2}{\pi N} xyz \frac{\ln(1+\nu)}{\nu} - \frac{5}{2\pi N}\frac{xyz}{(1+\nu)} + \frac{3}{\pi N}\frac{xy\sqrt{yz}}{(1+\nu)} , \tag{5.2b}$$

$$\beta_z = \frac{xyz}{\pi N}\frac{2}{(1+\nu)} + \frac{2xz^2}{\pi N}\left[\frac{5}{4(1+\nu)} + \frac{1}{\nu}\ln(1+\nu)\right] + \frac{2}{\pi N} xz\sqrt{yz}\frac{\ln(1+\nu)}{\nu} , \tag{5.2c}$$

$$\beta_\nu = \frac{\nu}{\pi N}\left[x + xz\left[\frac{-3}{4(1+\nu)} + \frac{\ln(1+\nu)}{2\nu}\right] + xy\left\{\frac{1}{4(1+\nu)} + \frac{5\ln(1+\nu)}{2\nu} + \frac{1}{\nu}\left[1-\left[\frac{1+\nu}{\nu}\right]\ln(1+\nu)\right]\right\} - x\sqrt{yz}\frac{\ln(1+\nu)}{\nu}\right] . \tag{5.2d}$$

These renormalization-group equations have to be solved with the initial condition,

$$y_0 = z_0 = \nu_0 = \frac{\rho g}{\lambda_0^2} . \tag{5.3}$$

The renormalization-group equations (5.2a)–(5.2d) are rather formidable. We will not attempt to solve them in their full complexity. Instead we consider the somewhat simpler limit of ν small, an assumption that will be shown to be consistent.

In the *small-ν limit*, the renormalization-group equations take the following forms:

$$\beta_x = -\frac{\epsilon x}{2} + \frac{x^2}{2\pi N} + \frac{x^2 y}{4\pi N} - \frac{x^2 z}{4\pi N} , \tag{5.4a}$$

$$\beta_y = -\frac{xyz}{2\pi N} + \frac{3}{\pi N} xy\sqrt{yz} , \tag{5.4b}$$

$$\beta_z = +\frac{2}{\pi N} xyz + \frac{9xz^2}{2\pi N} + \frac{2}{\pi N} xz\sqrt{yz} , \tag{5.4c}$$

$$\beta_\nu = \frac{\nu}{\pi N}\left[x - \frac{xz}{4} + \frac{9xy}{4} - x\sqrt{yz}\right] . \tag{5.4d}$$

In addition to a renormalization of x, y, z, and ν it is also necessary to renormalize the diffusion time τ and the Fermi velocity v_F [cf. Eqs. (4.4a) and (4.4b)].

They obey the renormalization-group equations,

$$\beta_\tau = -E_c \left. \frac{\partial \tau}{\partial E_c} \right|_R = \tau\left[\frac{\partial \ln Z_\tau}{\partial \ln E_c} - \frac{\partial \ln Z_\psi}{\partial \ln E_c}\right] \tag{5.5a}$$

and

$$\beta_{v_F} = v_F\left[\frac{\partial \ln Z_{v_F}}{\partial \ln E_c} + \frac{\partial \ln Z_\psi}{\partial \ln E_c}\right] . \tag{5.5b}$$

B. Solution of the renormalization-group equations ($d > 2$)

In order to solve the equations it is convenient to perform a change of variables. We define u and w as

$$u = xz , \quad w = xy . \tag{5.6}$$

That this set of variables (i.e., x, u, and w) is natural can be seen from the perturbative expressions of Sec. III.

The β functions, to order ϵ, are

$$\beta_x = -\frac{\epsilon}{2}x + \frac{x^2}{2\pi N} + \frac{xw}{4\pi N} - \frac{xu}{4\pi N} , \tag{5.7a}$$

$$\beta_u = -\frac{\epsilon}{2}u + \frac{xu}{2\pi N} + \frac{9uw}{4\pi N} + \frac{17u^2}{4\pi N} + \frac{2}{\pi N} u\sqrt{uw} , \tag{5.7b}$$

$$\beta_w = -\frac{\epsilon}{2}w - \frac{3uw}{4\pi N} + \frac{xw}{2\pi N} + \frac{w^2}{4\pi N} + \frac{3}{\pi N} w\sqrt{uw} , \tag{5.7c}$$

$$\beta_\nu = \frac{\nu}{\pi N}\left[x - \frac{u}{4} + \frac{9}{4}w - \sqrt{uw}\right] , \tag{5.7d}$$

with the initial condition $u_0 = w_0$. Thus if ν is small the RG equation for ν decouples from the other three variables.

When $u = w = \nu = 0$ these equations reduce to

$$\beta_x = -\frac{\epsilon}{2}x + \frac{x^2}{2\pi N} , \tag{5.8}$$

which is the RG equation of the nonintegrating theory.[15–19]

We now look for the fixed points of (5.7). They are ($d > 2$) as follows.

(i) *Trivial fixed point.* $x = u = w = 0$. It is *stable.*

(ii) *Critical fixed point.* $x^* = w^* = 0$, $u^* = (2\pi N\epsilon)/17$.

The linearized RG equations are

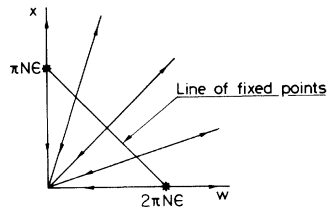

FIG. 17. RG flow in the basal plane $u=0$.

$$\beta_{u'}=\frac{\epsilon}{2}u'+\cdots\ , \tag{5.9a}$$

$$\beta_{x'}=-\tfrac{9}{17}\epsilon x'+\cdots\ , \tag{5.9b}$$

$$\beta_{w'}=-\tfrac{10}{17}\epsilon w'+\cdots\ , \tag{5.9c}$$

$$\beta_{\nu}=-\frac{\epsilon}{34}\nu+\cdots\ , \tag{5.9d}$$

where $x'=x$, $w'=w$, and $u'=u^*+u$. Thus it has one relevant eigenvalue $(+\epsilon/2)$ and two irrelevant ones. Hence it is, in fact, a critical fixed point. In addition, the departure from $\nu=0$ is *irrelevant* as shown in (5.9d).

(iii) *A line of unstable fixed points.* It lies in the $u=0$ plane and ranges from the Anderson fixed point $x^*=\pi N\epsilon$ and $w^*=0$ to a fixed point at $x=0$, $w^*=2\pi N\epsilon$.

The equation of this line is

$$\pi N\epsilon=x+\frac{w}{2}\ . \tag{5.10}$$

On the basal plane, $u=0$, the trajectories are straight lines,

$$w=kx\ , \tag{5.11}$$

as can be found by inspection. The flow is shown in Fig. 17.

At the Anderson fixed point $(w=0,x^*=\pi N\epsilon)$ the coupling constant u is marginally relevant,

$$\beta_u\approx\frac{17u^2}{4\pi N}+\cdots\ . \tag{5.12}$$

Everywhere else on the line of fixed points, the coupling constant u is relevant,

$$\beta_u\approx 2(1-c)\epsilon u+\cdots\ , \tag{5.13}$$

where $|c|\leq 1$ and $c=1$ when $w=0$, and $c=-1$ when $x=0$.

(iv) The line of fixed points of (iii) is unobservable with the exception of the Anderson fixed point. The reason is that the full flow of (5.7a)–(5.7d) is not physically relevant but only that part which is accessible from the initial condition,

$$u_0=w_0\ . \tag{5.14}$$

There is a critical surface that stands on the line of unstable critical points and flows into the critical fixed point of (iii).

The flow is shown in Fig. 18. The intersection of the

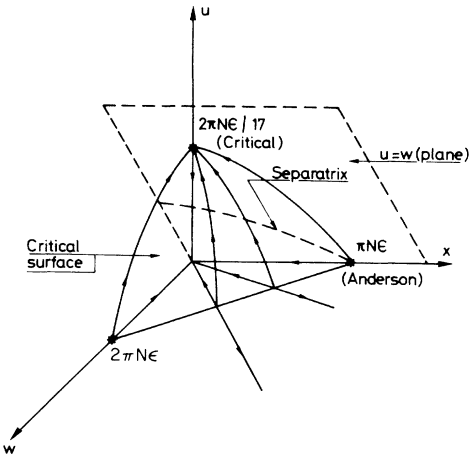

FIG. 18. Renormalization-group flows. The critical surface, the initial $u=w$ plane, the separatrix, and the Anderson and critical fixed points are shown.

plane $u=w$ (the initial condition) with the critical surface (i.e., the set of points that flow into the critical fixed point) defines the separatrix which locates the phase transition. All points *inside* the separatrix (i.e., *under* the critical surface) flow into the trivial fixed point. They belong to the metallic phase. All points *outside* the separatrix (i.e., *above* the critical surface) flow in infinity. They thus belong to the insulating (localized) phase of the theory.

(v) The "screening parameter" ν is found to be *irrelevant* at the critical fixed point,

$$\beta_{\nu}\approx-\frac{\epsilon}{34}\nu\ . \tag{5.15}$$

Thus the assumption of ν being small is consistent.

C. Exponent for the localization length ($d>2$)

In Sec. V B we showed that, to order ϵ, there is a critical fixed point that controls the long-distance properties of the theory. This fixed point was shown to have *two irrelevant* directions [cf. Eqs. (5.9b) and (5.9c)] and a *relevant* one [Eq. (5.9a)] with an eigenvalue equal to $\epsilon/2$. This is precisely the same eigenvalue found in the noninteracting theory at the Anderson fixed point $(w=u=0$, $x^*=\pi N\epsilon)$ and it is consistent with Wegner's scaling arguments.[22] To leading order in ϵ this result is then expected. It is known, however, that in the noninteracting theory, the two-loop contribution vanishes[16] and (perhaps) there are no further corrections to the eigenvalue. It would then be very interesting to perform a calculation of the RG equations to the next order (i.e., $1/N^2$) and see if there is a nonvanishing contribution to the eigenvalue due to interaction effects. Since the eigenvalue is the same as in the noninteracting theory, the localization length exponent ν (not to be confused with the screening parameter) remains equal to $1/\epsilon$.

D. dc conductivity

The dc conductivity can be calculated using linear-response theory. The Einstein relation, valid also when intersections are present, reads

$$\sigma \sim \frac{\partial n}{\partial \mu} \bar{D} , \tag{5.16}$$

where $\bar{D}$ is the diffusion constant and n the particle density. In the theory that we are presenting in this paper there is a parameter, the coefficient of q^2 in the diffusive mode, which we have called D, stressing the fact that D, scales like the diffusion constant. This parameter, however, coincides with the actual diffusion constant, calculated from linear-response theory, only in the noninteracting case. When interactions are on, the diffusive mode $K^{+-}(q,\omega)$ is a nontrivial function of Dq^2/ω. When one renormalizes the theory one is forced to choose a renormalization point which in fact is a choice of the ratio $(Dq^2)/\omega$. In this paper we have chosen to work in the dynamical regime $(Dq^2)/\omega \ll 1$ [as opposed to the static regime $(Dq^2)/\omega \gg 1$]. In the linear-response-theory calculation of $\bar{D}$, and hence of σ, *all the regimes contribute.* This is why, while our coefficient D and hence the coupling constant x are local quantities, the actual diffusion constant $\bar{D}$ is not. Indeed, a linear-response calculation of σ, to leading order in $1/N$, yields the result

$$\sigma_B(\omega)=\sigma_\infty+\sigma_\infty\left\{\frac{1}{N\pi}\left[\frac{1}{\nu}-\left[\frac{1+\nu}{\nu^2}\right]\ln(1+\nu)+\frac{\ln(1+\nu)}{\nu}\right]xz+\frac{x}{2\pi N}\right\}\ln\frac{\omega}{E_c} . \tag{5.17}$$

In the limit of small screening parameter, $\nu \to 0$, we find

$$\sigma(\omega)=\sigma_\infty+\sigma_\infty\frac{1}{2\pi N}(xz+x)\ln\frac{\omega}{E_c} . \tag{5.18}$$

We define the renormalized conductivity σ_R to be

$$\sigma_R=\sigma_B Z_\sigma , \tag{5.19}$$

where Z_σ, the renormalization constant for the conductivity, is equal to

$$Z_\sigma=1-\frac{1}{2\pi N}(xz+x)\ln\frac{\omega}{E_c} \equiv 1-\frac{1}{2\pi N}(u+x)\ln\frac{\omega}{E_c} . \tag{5.20}$$

In the course of renormalization physical, i.e., renormalized, quantities remain fixed. Thus we have

$$\frac{\partial \sigma_R}{\partial \ln E_c}=0 . \tag{5.21}$$

This condition yields a Callan-Symanzik-type equation at the fixed point

$$\left[E_c\frac{\partial}{\partial E_c}+\gamma_\sigma^*\right]\sigma_B=0 , \tag{5.22}$$

where the anomalous dimension γ_σ is given by (FP designates fixed point)

$$\gamma_\sigma^*=\left.\frac{\partial \ln Z_\sigma}{\partial \ln E_c}\right|_{\rm FP}=\frac{1}{2\pi N}(u^*+x^*) . \tag{5.23}$$

Thus

$$\gamma_\sigma^*=\frac{u^*}{2\pi N}=\frac{\epsilon}{17} . \tag{5.24}$$

Solving this equation, we get a connection between $\sigma(\omega)$ at two values of the cutoff, E_c and λE_c $(\lambda>1)$, as follows:

$$\sigma_B(\omega,\lambda E_c)\lambda^{\gamma_\sigma^*}=\sigma_B(\omega,E_c) . \tag{5.25}$$

However,

$$\sigma_B(\omega,\lambda E_c)=\sigma_B\left[\frac{\omega}{\lambda},E_c\right] . \tag{5.26}$$

Thus,

$$\sigma_B(\omega,E_c)=\lambda^{\gamma_\sigma^*}\sigma_B\left[\frac{\omega}{\lambda},E_c\right] . \tag{5.27}$$

Choosing $\lambda=\omega/\omega^*>1$, with ω^* some fixed frequency, we find

$$\sigma_B(\omega,E_c)=\left[\frac{\omega}{\omega^*}\right]^{\gamma_\sigma^*}\sigma_B(\omega^*,E_c) . \tag{5.28}$$

This equation implies that the conductivity, at the mobility edge, vanishes as a function of ω with an exponent s equal to γ_σ^*,

$$\sigma_B(\omega,E_c)=\text{const}\ \omega^s , \tag{5.29}$$

where

$$s=\gamma_\sigma^*=\frac{\epsilon}{17}$$

to leading order in ϵ.

We can also compute the behavior of the dc conductivity as the mobility edge is approached. The Callan-Symanzik equation is

$$\left[E_c\frac{\partial}{\partial E_c}-\beta_u\frac{\partial}{\partial u}+\gamma_\sigma\right]\sigma_B=0 , \tag{5.30}$$

where we have kept the relevant operator u only. By solving this equation we get

$$\sigma_{\rm dc}^B(u)=\sigma_{\rm dc}^B(u_0)\exp\left[-\int_{u_0}^{u}du'\frac{\gamma_\sigma(u')}{\beta_u(u')}\right]\times\Phi\left[\ln E_c+\int_{u_0}^{u}\frac{du'}{\beta_u(u')}\right] , \tag{5.31}$$

where Φ is a dimensionless function and u_0 is the initial iteration point. The singular part of $\sigma_{\rm dc}^B$ is then equal to

$$\sigma_{\rm dc}^B(\Delta)\approx\text{const}\ |\Delta|^t , \tag{5.32}$$

where

$$|\Delta| = \left| \frac{u-u^*}{u^*} \right|$$

and the exponent t is

$$t = \gamma_\sigma^* \Big/ \left[\frac{\partial \beta_u}{\partial u} \bigg|_{u^*} \right] = \frac{u^*}{2\pi N} \frac{2}{\epsilon} \equiv \nu s \ . \tag{5.33}$$

Explicitly we get $t = \frac{2}{17}$.

E. Scaling properties of the density of states at the Fermi surface

The (renormalized) density of states $N_R(E)$, at energy E, is related to the (renormalized) one-particle Green's function through the standard formula,

$$N_R(E) = \frac{1}{\pi} \mathrm{Im} G_R(\vec{r},\vec{r};E) \ . \tag{5.34}$$

In Sec. IV A we saw that in order to control the infrared divergencies that appear in the one-particle Green's function, to order $1/N$, it is necessary to renormalize the wave function as well as the Fermi velocity and the mean free time. These local quantities acquire a scale dependence in the process of renormalization. The density of states $N_R(E)$ is not a local quantity. As a matter of fact, there are contributions to $N_R(E)$ from energy scales *both small and large compared with* $1/\tau$. Thus a renormalization prescription of the type presented in Sec. IV A, where we set all energies to be either at the Fermi surface or much closer to it than $1/\tau$, is not appropriate for renormalizing the density of states. The situation is very similar to what we discussed in the preceding section concerning the conductivity.

An explicit computation of the density of states, to order $1/N$, yields the result

$$N_B(\omega) = N_\infty(\omega) \left[1 + \frac{1}{2\pi N} xz \frac{\ln(1+\nu)}{\nu} \ln \frac{\omega}{E_c} \right] . \tag{5.35}$$

The renormalized and bare density of states are connected through

$$N_R(\omega) = Z_N N_B(\omega) \ , \tag{5.36}$$

where the renormalization constant Z_N is equal to

$$Z_N = 1 - \frac{xz}{2\pi N} \ln \frac{\omega}{E_c} \equiv 1 - \frac{u}{2\pi N} \ln \frac{\omega}{E_c} \ . \tag{5.37}$$

Since Z_N and Z_σ coincide at the nontrivial fixed point, to this order on $1/N$, the exponents are the same.

Hence, at the mobility edge, the density of states near the Fermi surface vanishes like

$$N_B(\omega) = \text{const } \omega^\theta \ , \tag{5.38}$$

with $\theta = \epsilon/17$. The density of states at the Fermi surface vanishes, as the mobility edge is crossed, like

$$N_B(E_F,\Delta) = \text{const } |\Delta|^\delta \ , \tag{5.39}$$

where

$$\delta = \theta\nu = \tfrac{2}{17} \ .$$

These results agree with the work of Altshuler *et al.*[4] who first suggested that, at the mobility edge, the density of states may vanish as the Fermi surface is approached.

F. Scaling behavior of the mean free time and the Fermi velocity

Equations (5.5a) and (5.5b) govern the scaling properties of the (bare) mean free time τ and Fermi velocity v_F. The solution of (5.5a) and (5.5b) yields the cutoff dependence in τ and v_F. A calculation in the same spirit of what was done for the density of states and the conductivity yields the following results:

(i) *Scaling of* τ. At the fixed point the (frequency-dependent) mean free time is found to vanish like

$$\tau(E,E_c) \approx \left[\frac{E}{E^*} \right]^\lambda \tau(E^*,E_c) \ , \tag{5.40}$$

with an exponent λ, to order ϵ, equal to

$$\lambda = \beta_\tau'(u^*) = \tfrac{7}{34}\epsilon \ . \tag{5.41}$$

The mean free time *at the Fermi surface* is also found to vanish as the mobility edge is crossed,

$$\tau(E_F) \sim \text{const } |\Delta|^\zeta \ , \tag{5.42}$$

where the exponent ζ, to order ϵ, is

$$\zeta = \nu\lambda = \tfrac{7}{17} \ . \tag{5.43}$$

(ii) *Scaling of the Fermi velocity.* The (frequency-dependent) Fermi velocity is found to diverge like

$$v_F(E,E_c) \sim \left[\frac{E}{E^*} \right]^{-\sigma} v_F(E^*,E_c) \ , \tag{5.44}$$

where the exponent σ is found, to order ϵ, to be

$$\sigma = \tfrac{5}{34}\epsilon \ .$$

The Fermi velocity, at the Fermi surface, diverges as the velocity edge is crossed like

$$v_F(E_F) \sim \text{const } |\Delta|^{-\rho} \ , \tag{5.45}$$

where, to order ϵ, ρ is found to be

$$\rho = \sigma\nu = \tfrac{5}{17} \ . \tag{5.46}$$

The vanishing of the mean free time and the divergence of the Fermi velocity signal the breakdown of the Landau theory of the Fermi liquid near the transition.[23]

G. The exponent η

We have already noted that the triviality of the anomalous dimension exponent η in the noninteracting theory no longer holds when interactions are considered. We now substantiate our claim with an explicit calculation of η.

The renormalization of the diffusive mode K^{+-} has been given in Eq. (4.9a). Define an effective renormalization constant Z_K by

$$Z_K = Z_\tau Z_\psi Z_Q^{-1} , \tag{5.47}$$

and an anomalous dimension γ_K,

$$\gamma_K = -E_c \frac{\partial}{\partial E_c} \ln Z_K . \tag{5.48}$$

At criticality γ_K assumes its fixed-point value,

$$\gamma_K^* = \eta = \frac{u^*}{4\pi N} = \frac{\epsilon}{34} , \tag{5.49}$$

which defines the exponent η.

At the fixed point the bare diffusive mode obeys the renormalization-group equation

$$\left[E_c \frac{\partial}{\partial E_c} - \gamma_K^* \right] K_B^{-1} = 0 . \tag{5.50}$$

Thus, the diffusive mode, at zero momentum, scales like

$$K_B^{-1}(E) \sim \left[\frac{E}{E^*} \right]^{1-\eta} K_B^{-1}(E^*) \tag{5.51}$$

at the transition. The exponent η is given by Eq. (5.43).

H. The insulating phase

So far we have considered the behavior of the system either in the metallic state or at criticality. In the localized phase the situation is more complicated.

First of all, our methods do not allow the study of the system for strong disorder and strong interactions. We can only see a runaway behavior. If we assume that physics is smooth between the fixed point of order ϵ and the limiting behavior for $(x,u,w) \to \infty$ then we can use the renormalization group to match. However, this requires a knowledge of the properties of the phase (or phases) that may exist in this limit.

The physics in the strong-coupling–strong-disorder limit was first considered by Efros and Shklovskii[24] and more recently by Davies, Lee, and Rice.[25] They both study the case of Coulomb (i.e., long-range) interactions although the results are qualitatively the same for short-range interactions. The insulating phase can be defined properly only if the system is placed on a lattice. In this case both groups of authors[24,25] find that the density of states is

$$N(E) \sim \exp\left[-\left[\frac{E_0}{E} \right]^{1/2} \right] \tag{5.52}$$

in three dimensions while the behavior in two dimensions is somewhat unclear. In any event an energy scale E_0 seems to exist below which the density of states drops dramatically to zero. This scale can be found by means of the renormalization group since it obeys

$$E_c \frac{\partial E_0}{\partial E_c} = 0 . \tag{5.53}$$

Define a (dimensionless) function F of the scaling parameters. In terms of F, E_0 is

$$E_0 = E_c F(x,u,w) . \tag{5.54}$$

The function F obeys the RG equation

$$\left[1 - \beta_x \frac{\partial}{\partial_x} - \beta_u \frac{\partial}{\partial u} - \beta_w \frac{\partial}{\partial w} \right] F = 0 . \tag{5.55}$$

By solving (5.49) along the relevant trajectory we find that this scale vanishes near the phase transition like

$$E_0 \sim \text{const} \; |\Delta|^{2/\epsilon} \tag{5.56}$$

for $d = 2+\epsilon$. This result is consistent with the localization length diverging like $|\Delta|^{-1/\epsilon}$.

What is the meaning of this scale? If we place the system on a lattice then the "insulating phase" is characterized by a reduced Hamiltonian in which the kinetic energy term is deleted. Davies, Lee, and Rice[25] have shown that, for $N = 1$, the Hamiltonian reduces to that of the classical Ising antiferromagnet, in our case with nearest-neighbor interactions, in a random magnetic field. General arguments of the type devised by Emery[26] allows one to incorporate quantum fluctuations within degenerate perturbation theory. The effective Hamiltonian is then a quantum *XXZ* antiferromagnet in a quenched random magnetic field pointing in the Z direction. Very little is known about the properties of such systems. But qualitatively one expects to find two insulating phases: (a) an "Anderson insulator" without a gap but with all states localized and (b) a "deformed Wigner crystal," or charged-density wave (the analog of the antiferromagnet phase), which has a gap in the single-particle spectrum. In both cases an energy scale exists and by scaling must behave like (5.56). From what it is known of magnets in random fields one may expect that, *at least in the classical limit* and for space dimensions up to a lower critical dimension d_c, which could be two[27] or three,[28] the antiferromagnetic phase might be absent. Clearly these issues require a more detailed study and are not well understood.

In two-space dimensions the metallic phase is absent. The RG trajectories run away to the strong-coupling–strong-disorder limit. The system is an insulator and the localization length is finite. For weak coupling and weak disorder the localization length diverges and the energy scale vanishes with an essential singularity

$$E_0 \sim \text{const} \exp\left[-\frac{2\pi N}{u_0} \right] . \tag{5.57}$$

VI. CONCLUSION

We have shown in this paper that a systematic expansion around a weakly disordered metallic phase can be done by means of the $1/N$ expansion. This expansion has to be renormalized since divergencies are present at every order in $1/N$. We have not attempted here to prove renormalizability, i.e., the existence of only a *finite* number of *relevant* operators and renormalization constants. We argue, however, that if the one-particle Green's function has singularities *only* of the form $(\Omega/E_c)^p \ln^q(\Omega/E_c)$, then the

Ward identities do imply the existence of just a finite number of relevant operators. This point *does* need to be checked further by means of a two-loop (i.e., $1/N^2$) calculation.

The renormalization-group equations presented here, and the phase diagram that derives from it, do seem to have all the right physical properties. To improve over this calculation it is important to have a better control on the interacting fixed point by considering more realistic interactions. This will make possible a better understanding of the nature of the insulating phase. Also the role of long-range interactions, which are technically more difficult although physically more relevant, has to be considered. We will return to these problems in another communication.

Note added. After this work was completed we received a copy of unpublished work by G. Grest and P. A. Lee,[29] where they propose a scaling theory for a disordered interacting electron gas based on a second-order calculation of the conductivity, density of states, and spin susceptibility. Their approach, based on linear-response theory, differs from ours and so do the results. We also became aware of unpublished works by A. M. Finkelstein, who proposes a similar theory in the presence of a magnetic field. The connection between both papers and our work is currently under study. A preliminary report of this work was presented elsewhere.[30]

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