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Elastic Model of Dry Friction[1]

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Abstract—Friction of elastic bodies is connected with the passing through the metastable states that arise at the contact of surfaces rubbing against each other. Three models are considered that give rise to the metastable states. Friction forces and their dependence on the pressure are calculated. In Appendix A, the contact problem of elasticity theory is solved with adhesion taken into account.

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1. INTRODUCTION

In the process of friction, when one rough surface rubs against another, friction forces arise where surfaces come into contact. Therefore, there are two kinds of problems connected with this phenomenon. The first is, which processes occur at the contact of two surfaces; and the second, in what way do the random forces arising at different points of contact add up to the total friction force? In this paper, deformations at the contact points are assumed to be elastic.[2]

If the deformation of two contacting surfaces is elastic, then the surfaces are not changed by friction. The work of friction forces hence results in radiation of sound waves. The energy of these sound waves dissipates in the bulk of the rubbing bodies. The power spent for the radiation of sound is proportional to acceleration squared. When the relative velocities of the rubbing bodies are small, a large acceleration, independent of this velocity, arises only when the contact turns from a metastable state into a stable one. The aim of this paper is to elucidate the problem of how metastable sates arise in an elastic medium. Two possibilities arise in this case: the metastability arises in each contact separately, or the region of the surface containing a large number of contacts transfers into a metastable state.

The problem of reaction of an elastic structure with randomly arranged defects has been considered previously in connection with pinning of vortex lines in superconductors [3–5]. The method developed in these papers is applicable to the problem of dry friction.

This paper is arranged as follows. In Sections 2 and 3, the case of individual pinning is considered, where a separate contact can be in a metastable state. In Section 2, the case is studied where the metastability is caused by adhesion of contacts caused by intermolecular interaction. In Section 3, we examine the model of thin and long contacts (the "brush" model). In this case, the metastability arises due to the loss of stability. Section 4 is devoted to the study of collective pinning, when the metastability of the surface containing a large number of contacts is in a metastable state. Details of calculations related to each section are presented in the appendices.

2. ADHESION

The contact of rubbing surfaces is usually realized on a small area and the rough points touch one another. Therefore, a good model of a rough surface is that of balls randomly scattered on the surface. The radii of the balls R are assumed to be equal. We first consider the strictly elastic case, where the forces arising at the contact of the balls are uniquely defined by their mutual location and the elastic properties of solids. These forces are directed normally to the contact areas of the balls, and therefore have components tangential to the surface, averaged over roughness. The sum of these forces is equal to the total normal force N. The directions of tangential forces are determined by the mutual arrangement of the balls, and the total tangential force is zero, because of the randomness of the arrangement. Therefore, in the purely elastic case, the friction force is zero.

A nonzero friction force can arise if, besides the elastic forces, an adhesion due to molecular interaction of two contacting bodies is taken into account. In this case, the deformation of two contacting bodies is a multivalued function of their mutual position, and therefore the forces are also multivalued functions. Hence, the forces arising at contacts depend not only on the current arrangement of two rubbing bodies but also on the prehistory of the formation of this arrange-

[1] The article is published in the original.

[2] It is believed [1, 2] that at low pressure, the friction force is determined mainly by intermolecular forces, while at high pressure, a plastic deformation (ploughing) is more essential.

ment. If the balls are coming into contact, the force is zero until they touch each other. If the balls are separating, they remain in contact and the force is not zero even when the distance between their centres exceeds the sum of the radii. Therefore, at a relative shift of the surfaces, there arises an averaged force, whose direction is opposite to the shift.

To calculate the friction coefficient, it is convenient to use the energy consideration. The work of friction forces calculated per one contact is equal to the energy required for breaking the balls in contact plus the energy of the balls in contact that have only once touched one another. This energy is calculated in Appendix A (Eq. (30)):

$$\tilde{\mathscr{E}} = 1.05(\pi\alpha)^{5/3}\left(\frac{R^2}{E}\right)^{2/3}. \tag{1}$$

The friction force depends on the distribution of heights x of the tops perpendicular to the surface of sliding. If $C_{1,2}(x)$ is the number of tops per unit area of the first (second) body in a unit range of x, then, as one body shifts along another, the number of contacts per unit of length is

$$N(h) = S\int_0^h dx_1 \int_0^{h-x_1} dx_2 \times C_1(x_1)C_2(x_2)a(h-x_1-x_2), \tag{2}$$

where $a(h) = \sqrt{Rh}$ and S is the contact surface area. Taking into account that the energy $\tilde{\mathscr{E}}$ is dissipated at any contact, we obtain the friction force

$$f_{\rm fr} = \frac{\tilde{\mathscr{E}}N(h)}{S}. \tag{3}$$

To determine the friction coefficient, the friction force $f_{\rm fr}$ should be divided by the pressure p; besides, the value of the vertical shift h of the rubbing bodies should be determined through the pressure. The energy dissipated per unit area is

$$\int_0^h dx_1 C_1(x_1) \int_0^{h-x_1} dx_2 C_2(x_2) \int 2\pi\rho d\rho \mathscr{E}(\xi), \quad \xi = \left(h - x_1 - x_2 - \frac{\rho^2}{2R}\right), \tag{4}$$

where $\mathscr{E}(h)$ is the energy at a single contact as a function of the vertical shift h. Calculated in Appendix A, $\mathscr{E}(h)$ is a multivalued function that has a jump at $h = 0$ and at $h = h_{\rm min}$. The energy in (4) includes the work of forces of normal pressure and the energy jumps. In order to calculate the pressure p, it is necessary to differentiate the continuous part of energy (4) with respect to h:

$$p = 2\pi R\int_0^h dx_1 C_1(x_1) \int_0^{h-x_1} dx_2 C_2(x_2) \times \{\mathscr{E}(h-x_1-x_2) - \Delta\mathscr{E}\}, \tag{5}$$

where $\Delta\mathscr{E}$ is the energy jump at the transition from the metastable state and depends on prehistory. If the pressure increases, then $\Delta\mathscr{E} = \mathscr{E}(h = 0)$. If it decreases, then $\Delta\mathscr{E} = \mathscr{E}(h_{\rm min})$. When the motion proceeds under a constant pressure, the number of converging contacts is equal to the number of diverging contacts, and therefore $\Delta\mathscr{E} = (\mathscr{E}(h_{\rm min}) + \mathscr{E}(h = 0))/2$. Hence, the pressure p is a multivalued function of h. In any case, a finite value $h \neq 0$ and, consequently, a finite friction force correspond to zero pressure $p = 0$. We suppose that

$$C(x) = C\left(\frac{x}{\langle x\rangle}\right)^{\nu}.$$

Then the friction force, by the order of magnitude, is

$$f_{\rm fr} \approx \tilde{\mathscr{E}}a(h)C_1(h)C_2(h) \sim \alpha\left(\frac{\alpha}{E}\right)^{(4\nu+7)/3} \times \frac{C_1C_2R^{(2\nu+8)/3}}{\langle x\rangle^{2\nu+2}}. \tag{6}$$

At $p = 0$, the friction force strongly depends on the concentration of the contacts and their spread in heights. If there is no spread ($\nu = -1$), then

$$f_{\rm fr} \approx \frac{\alpha^2}{E}C_1C_2R^2. \tag{7}$$

If the pressure is sufficiently high,

$$p \gg (\alpha^5R^7)^{1/3}C_1C_2E^{-2/3}\left(\frac{\alpha^2R}{E\langle x\rangle^3}\right)^{(2\nu+2)/3},$$

then the dependences $\mathscr{E}(h)$ and $a(h)$ can be determined without taking adhesion into account:

$$k_{\rm fr} = \frac{f_{\rm fr}}{p} \sim \left(R\frac{\alpha^2}{E^2}\right)^{1/3} \times \left\{\frac{C_1C_2R^{(3/2)}}{\langle x\rangle^{2\nu+2}}\right\}^{4/(4\nu+9)} \left(\frac{E}{p}\right)^{4/(4\nu+9)}. \tag{8}$$

As can be seen from Eq. (8), the friction coefficient $k_{\rm fr}$ is pressure dependent. If the experimental value $\nu = 2$ is assumed, then

$$k_{\rm fr} \sim p^{-4/17}.$$

3. THE "BRUSH"

In this section, we consider a model in which metastable states of contacts arise when adhesion is not taken into account. We suppose that the surface of one of the bodies has noncompressible roughness with the rounding–off radius R. Another surface resembles a brush. It can be viewed as a rigid plate with elastic rods emerging out of the plate. The length of each rod is l and the area of its cross section is S. For convenience of calculation, we assume that $l \ll R$.

The successive arrangement of the rods in plates passing from the left to the right is shown in Fig. la. This figure shows that at the right slope of the roughness, there are metastable states at which the rod is bent to the left. When the plate moves along the right slope, the normal force F is developed. When this force slightly exceeds the Euler instability threshold $F_E = \pi^2 EI/4l^2$ (where I is moment of inertia of the rod cross section), the rod jumps to the bend on the right (dashed line in Fig. la). Elastic energies corresponding to these two positions of the rod are different and this difference is transferred into heat. The difference of elastic energies $\tilde{\mathscr{E}}$ is calculated in Appendix B. It is equal to

$$\tilde{\mathscr{E}} = 4SEh\left[\frac{4Slh}{\pi^2 I} - 1\right]. \quad (9)$$

The friction coefficient can be found using Eqs. (3) and (5). For simplicity, we assume that $C_{1,2}(x) = C_{1,2}\delta(x)$. As a result, obtain the following expression for the friction coefficient:

$$k_{\rm fr} = A\left(\frac{p}{E}\right)^{1/4}\frac{p - p_c}{p}, \quad (10)$$

$$p_c = \frac{\pi^2}{12}E\frac{IRC_1C_2}{Sl^2},$$
$$A = 2\left(\frac{3^5}{\pi^{10}}\right)^{1/4}\frac{S}{E}\left(\frac{l^9}{C_1C_2RS}\right)^{1/4}. \quad (11)$$

It is important in deriving Eqs. (10) and (11) that some positions of the fixed end of the rod correspond to the position of a free rod. One of this positions is realized in motion from left to right, and the other, in motion from right to left. Quite a different picture is realized at low pressures (see Fig. 1b), when the normal force does not exceed the instability threshold F_E. In this case, the elastic forces acting in the contact at symmetric points of the roughness have equal and oppositely directed tangential components. In calculating the total force, we should average over possible positions of the fixed end of the rod. Such an averaging describes both the sum of the forces arising at different moments of the rod motion and the sum of the forces acting at the system of randomly arranged rods at rest. If such an averaging is performed in the situation shown in Fig. 1b, the resulting force is zero. Therefore, the friction force does not arise in the "brush" model at low pressure.

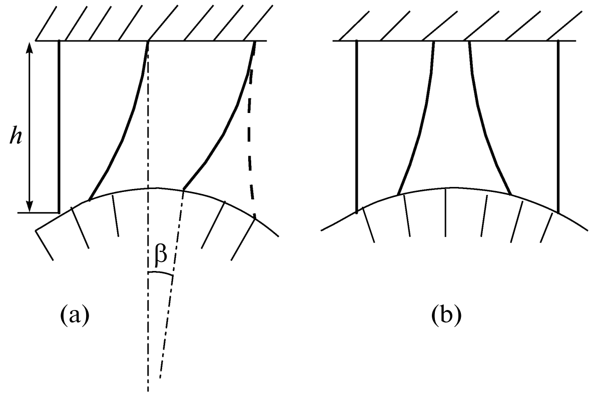

Fig. 1.

4. COLLECTIVE PINNING

In this section, we consider contact of two rough surfaces similar to those considered in Section 2. The difference is that we here neglect the adhesion.

We assume that there is no metastable states in each individual contact. The tangential forces $\mathbf{f}_i$, acting at the points of contact can be assumed to be random and to depend on the position of the contact:

$$\langle \mathbf{f}_i \rangle = 0, \quad \langle f_i^\alpha f_j^\beta \rangle = \langle f^2 \rangle \delta^{\alpha\beta}\delta_{ij}.$$

If the mutual influence of contacts is not taken into consideration, the mean tangential force $f_{\rm fr}$ is

$$f_{\rm fr} \sim \sqrt{N\langle f^2 \rangle},$$

where N the number of contacts. Hence, the friction force is proportional to the root of the area S, and the friction coefficient $k_{\rm fr}$ is inversely proportional to $\sqrt{S}$:

$$k_{\rm fr} \sim \frac{1}{L},$$

where L is the linear size of the surface if contact of two rubbing bodies.

If the mutual influence of contacts is taken into account, then it turns out that for a sufficiently large surface, the friction coefficient is independent of its area.

To determine the deformation arising due to the force acting at the points of contact of the rubbing surfaces, we must find the Green's function of the elasticity theory. This Green's function depends on the ratio of compressibilities of the rubbing surfaces. For definiteness, we assume that the rubbing bodies have identical elastic properties (the Young moduli E and the Poisson coefficients σ). In this case, the Green's function can be found in the problem to Section 8 in [6]. The tangential displacement $\mathbf{u}_i$ of the ith contact is therefore related to the forces $\mathbf{f}_j$ acting at the jth contact as

$$\mathbf{u}_i = \sum_{j \neq i} \hat{G}(\mathbf{r}_{ij})\mathbf{f}(\mathbf{r}_j - \mathbf{u}_j), \quad (12)$$

$$G^{\alpha\beta}(\mathbf{r}) = \frac{1+\sigma}{8\pi E(1-\sigma)}\left\{\frac{3-4\sigma}{r}\delta^{\alpha\beta} + \frac{r^\alpha r^\beta}{r^3}\right\}. \quad (13)$$

Using Eq. (13), we estimate the mean square of displacement $\langle \mathbf{u}^2 \rangle$ due to the action of the forces applied at a large number of contacts:

$$\langle \mathbf{u}^2 \rangle = \sum_{jk} G(\mathbf{r}_j)G(\mathbf{r}_k)\langle f_j f_k \rangle \sim \sum_j G^2(\mathbf{r}_j)\langle f^2 \rangle \sim \frac{C\langle f^2 \rangle}{E}\ln\frac{L}{\rho}. \quad (14)$$

Here, L is the linear dimension of the area of rubbing surfaces. Thus, if this area is large, then the collective action of the forces applied to different contacts results in a large displacement even if the forces at each individual contact are small.

In the same way, we can calculate the mean square of the relative displacement at points $\mathbf{r}$ and $\mathbf{r}'$:

$$\langle [\mathbf{u}(\mathbf{r}) - \mathbf{u}(\mathbf{r}')]^2 \rangle \sim \frac{C_1\langle f^2 \rangle}{E^2}\ln\frac{|\mathbf{r}-\mathbf{r}'|}{\rho}.$$

It is clear from this equation that the relative displacement of distant points is large, although the solid bodies are rigid. Therefore, a large surface can be divided into areas of finite dimensions R_c such that relative displacements in one such region are less than or of the order of the size ρ of a single contact. The relative displacements of different regions are of the order of ρ or even larger. Each such region makes an independent contribution to the friction force, which is therefore proportional to the number of regions and hence to the total area. The friction coefficient $k_{\rm fr}$ in this case is the same as for a separate region and inversely proportional to R_c. The correlation length R_c can be estimated as

$$\ln\frac{R_c}{\rho} \approx \frac{E^2\rho^2}{C_1\langle f^2 \rangle^2}. \quad (15)$$

Therefore, the correlation length R_c depends exponentially on the pressure p. The exponential is determined in Appendix C. If the size R of the contact area of two rubbing bodies is small ($R \ll R_c$), then $f_{\rm fr} \sim R$. In the opposite limit $R \gg R_c$, the contribution of each region of size R_c is proportional to the correlation length. Multiplying by the number of such regions, we obtain the following estimate for the friction coefficient with exponential precision:

$$k_{\rm fr} \sim \exp\left[-\left(\frac{E}{p}\right)^{8/5}\right]. \quad (16)$$

5. CONCLUSION

The goal of this paper is mainly methodical. We tried to answer the question of how the irreversible energy dissipation can arise in reversible elasticity theory. For all the models considered, there can be only one answer: "the energy dissipates in transitions from a metastable state into a stable one." On the other hand, none of the proposed mechanisms yields the Coulomb–Amontons law (the friction force is proportional to normal pressure). Apparently, this means that while considering the contact of a large number of real bodies, it is impossible to use the elasticity theory, because the arising deformations are plastic. Another inelastic mechanism may be the breaking of parts of the body when it sticks to another body. This mechanism may be significantly weakened by applying the appropriate boundary lubrication.

Although inelastic mechanisms are important, the elastic mechanism of friction also exists. For the majority of bodies, it seems to produce a small contribution to the friction force. But in the case where the friction force is determined by the elastic mechanism, the friction coefficient strongly depends on pressure. This dependence is different in different cases. In the model with adhesion, the friction coefficient $k_{\rm fr}$ is inversely proportional to the pressure p ($k_{\rm fr} \sim A/p$ as $p \longrightarrow 0$). The coefficient A strongly depends on the dimensions of the surface roughness. In the "brush" model, the friction force arises under pressures exceeding the threshold values determined by the Euler instability. Besides the rough surfaces, this model may describe the phenomena occurring with the boundary lubrication, if the lubricant consists of long molecules that stick with their ends to one of the bodies in contact.

The collective pinning should be taken into account in cases where the friction force is small and other mechanisms do not lead to friction. In this case, for the surface of a characteristic size L, the friction coefficient is proportional to $1/L$. If this dimension is large, the friction coefficient does not depend on L and depends on pressure exponentially. Equations (15) and (16) therefore provide the least possible friction coefficient.

APPENDIX A

Contact Problem in the Elasticity Theory with for Adhesion

Hertz solved the problem of the contact of two elastic bodies (see [6]). He considered two balls with radii R_1 and R_2, with the Young moduli E_1 and E_2 and the Poisson coefficients σ_1 and σ_2. The balls are compressed by the force F. Such a problem turned out to be equivalent to the problem of the contact of a ball with the radius

$$R = \left[\frac{1}{R_1} + \frac{1}{R_2}\right]^{-1} \quad (17)$$

and the effective Young modulus

$$\Theta = \left[\frac{1-\sigma_1^2}{E_1} + \frac{1-\sigma_2^2}{E_2}\right]^{-1} \quad (18)$$

with a rigid plane. The size a of the contact area was chosen such that the stress on its boundaries vanished. The result is

$$a^2 = Rh, \quad (19)$$

where h is the maximum vertical shift.

In the Hertz problem, the elastic deformation energy $E(a, h)$ can be calculated with the given contact radius a and vertical deformation h. Adding the work of external forces to this energy and then minimizing the sum with respect to a and h, the dependences $a(h)$, $F(h)$, and $E(h)$ can be determined. In the case where adhesion is taken into account, the elastic deformation energy for the given a and h should be calculated and the work of external forces and the adhesion energy $\pi\alpha a^2$ should be added to it (α is the surface energy of adhesion). The total energy $E(a, h)$ determined in such a way should be minimized with respect to a, and $a(h)$ and $E(h)$ should be found.

To fulfil this program, we have to solve the following equation for the density of forces $P(r)$ at the contact area $r < a$:

$$h - \frac{r^2}{2R} = \Theta\int\frac{P(r_1)}{|r - r_1|}dr_1. \quad (20)$$

Equation (20) resembles the electrostatic relation that connects potential with the charge density. If a and h are connected by relation (19), then

$$P(r) \propto \sqrt{1 - \frac{r^2}{a^2}}. \quad (21)$$

In the general case, this solution can be written as a linear combination of two expressions

$$\sqrt{1 - \frac{r^2}{a^2}} \text{ and } \frac{1}{\sqrt{1 - r^2/a^2}}.$$

Evaluating the integrals (also see [7]) finally gives the expression

$$P(r) = \frac{2a\Theta}{\pi R}\sqrt{1 - \frac{r^2}{a^2}} + \frac{\Theta}{\pi a}\left(h - \frac{a^2}{R}\right)\frac{1}{\sqrt{1 - r^2/a^2}}. \quad (22)$$

The elastic force F is given by the integral over the contact area with the integrand $P(r)$:

$$F = \Theta\left[2ah - \frac{2}{3}\frac{a^3}{R}\right]. \quad (23)$$

The total energy includes the elastic deformation energy at $h = h_0 = a^2/R$, the work of the force F on the way from h_0 to h at a fixed a, and the adhesion energy $\pi\alpha a^2$:

$$E = (\pi\alpha)^{5/3}\left(\frac{R^2}{\Theta}\right)^{2/3}\mathcal{E}(x, y), \quad (24)$$

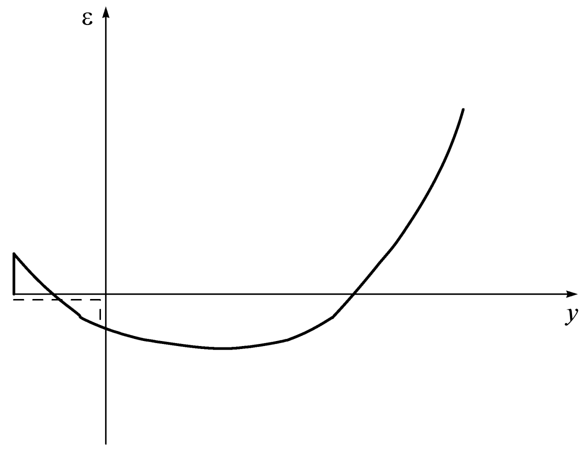

Fig. 2. Dimensionless energy $\mathcal{E}(x, y)$ under the condition $\partial\mathcal{E}/\partial x = 0$ as a function of y.

$$\mathcal{E}(x, y) = \frac{x^5}{5} - \frac{2}{3}x^3y + xy^2 - x^2, \quad (25)$$

where we introduce the dimensionless variables

$$x = a\left(\frac{\Theta}{\pi\alpha R^3}\right)^{1/3}, \quad y = h\left(\frac{\Theta^2}{\pi^2\alpha^2 R}\right). \quad (26)$$

The radius of the contact is determined from the energy minimum condition:

$$0 = \frac{\partial \mathcal{E}}{\partial x} = (x^2 - y)^2 - 2x. \quad (27)$$

If this last equation is solved and its solution $x(y)$ is substituted in the expression for $\mathcal{E}(x, y)$, then we obtain the y-dependence of the total energy. This dependence is plotted in Fig. 2. The solid curve shows the dependence $\mathcal{E}(y)$ under load, and the dotted line shows $\mathcal{E}(y)$ under compression. The minimum value of y compatible with the equilibrium condition corresponds to the breaking-off of the stuck ball:

$$y_{\min} = -0.5, \quad \mathcal{E}(y_{\min}) = 0.1. \quad (28)$$

At the instant of contact,

$$y = 0, \quad x = 2^{1/3}, \quad \mathcal{E}(0) = -3\frac{2^{1/3}}{5} = -0.95. \quad (29)$$

And finally, for the hysteresis energy, we find

$$E = 1.05(\pi\alpha)^{5/3}\left(\frac{R^2}{\Theta}\right)^{2/3}. \quad (30)$$

APPENDIX B

We consider the "brush" model in the case where the normal force F only slightly exceeds the Euler instability threshold:

$$F_E = \frac{\pi^2}{4}\frac{EI}{l^2}. \quad (31)$$

At the top of the roughness, the force is equal to (see Fig. la)

$$F = \frac{SEh}{l}. \tag{32}$$

The bending of the bar is determined by the angle θ_0, which can be found from the equation

$$l = \frac{\sqrt{EI}}{2F}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos(\theta-\beta)-\cos(\theta_0-\beta)}}. \tag{33}$$

Expanding the cosines in Eq. (33) in θ, θ_0, and β and the force F in $F - F_E$, we obtain

$$\theta_0^3 - 8\theta_0\frac{F-F_E}{F_E} - \frac{32\beta}{\pi} = 0. \tag{34}$$

The solution $\theta(\beta)$ of Eq. (34) is single—valued if

$$\beta > \beta_0 = \frac{\pi^2}{16}\left[\frac{8}{3}\left(\frac{4slh}{\pi^2 I} - 1\right)\right]^{3/2}. \tag{35}$$

At smaller values of β, Eq. (34) has three solutions, which correspond to a metastable, a stable, and an unstable state.

In deriving Eqs. (34) and (35), the terms of the order of $R\beta^2$ were neglected in comparison with h. This is justified if

$$\frac{4Slh}{\pi^2 I} \gg \left(\frac{192}{\pi^2}\frac{h}{R}\right)^{1/3}. \tag{36}$$

At $\beta = \beta_0$, Eq. (34) has two solutions

$$\begin{aligned}\theta_{1,2} &= (\pm 3 - 1)\sqrt{\frac{2}{3}\frac{F-F_E}{F_E}} \\ &= (\pm 3 - 1)\left\{\frac{2}{3}\left[\frac{4slh}{\pi^2 I} - 1\right]\right\}^{1/2}.\end{aligned} \tag{37}$$

The bending energy $\mathscr{E}$ is equal to the work of the force F:

$$\mathscr{E} = -Fl\{\cos(\theta-\beta) - 1\} \approx \frac{Fl\theta_0^2}{2}. \tag{38}$$

Therefore, the energy jump at the transition from the metastable state to the stable one is equal to

$$\tilde{\mathscr{E}} = \frac{32}{9}SEh\left[\frac{4Slh}{\pi^2 I} - 1\right]. \tag{39}$$

APPENDIX C

We here present the procedure that allows replacing a set of a large number of contacts by a single one.

The forces acting on such enlarged contacts increase, and large forces give rise to metastable states. In the metastable states, the derivatives of the force with respect to coordinates becomes infinite.

We begin with reproducing Eq. (12):

$$\mathbf{u}_i = \sum_{j\neq i}\hat{G}(\mathbf{r}_i - \mathbf{r}_j)\mathbf{f}(\mathbf{r}_j - \mathbf{u}_j). \tag{40}$$

Here, $\mathbf{r}_i$ is the position of a contact point under the condition that the mutual influence of the contact is not taken into account, and $\mathbf{u}_i$ is the deformation caused by such an influence. We seek the minimum of the energy

$$\mathscr{E} = \sum_i \mathscr{E}(\mathbf{r}_i - \mathbf{r}_j) + \frac{1}{2}\sum_{j\neq i}\mathbf{u}_i\hat{G}_{ij}\mathbf{u}_j. \tag{41}$$

The force $\mathbf{f}$ is linked to the energy $\mathscr{E}$ by the usual relation

$$\mathbf{f}(\mathbf{r}_i - \mathbf{u}_i) = \frac{\partial}{\partial \mathbf{u}_i}\mathscr{E}(\mathbf{r}_i - \mathbf{u}_i). \tag{42}$$

The mutual influence arises because the argument of the force $\mathbf{f}_i$ depends on the displacement $\mathbf{u}_i$, which is determined by forces acting at other contacts. The displacement $\mathbf{u}(\mathbf{r}_i)$ can be represented as the sum

$$\mathbf{u}_i = \mathbf{u}_i^{(1)} + \mathbf{w}_i^{(1)}, \tag{43}$$

$$\begin{aligned}\mathbf{w}_i^{(1)} &= \sum_{j\neq i}\hat{G}_{ij}\mathbf{f}(\mathbf{r}_j - \mathbf{u}_i^{(1)} - \mathbf{w}_i^{(1)}), \\ &|\mathbf{r}_i - \mathbf{r}_j| \le R_1.\end{aligned} \tag{44}$$

The first term in the sum (43) is equal to the displacement caused by the influence of the forces acting at distant contacts with $|\mathbf{r}_i - \mathbf{r}_j| > R_1$. The second term determines the displacement caused by the influence of the forces acting at the close contacts. If the displacement $\mathbf{w}$ determined by Eq. (44) is substituted in Eq. (41) for the elastic energy $\mathscr{E}$, the latter becomes

$$\mathscr{E} = \sum_i \mathscr{E}_R(\mathbf{r}_i - \mathbf{u}_i^{(1)}) + \frac{1}{2}\sum_{i\neq j}\mathbf{u}_i^{(1)}\hat{G}_{ij}\mathbf{u}_j^{(1)}, \tag{45}$$

$$\mathbf{f}_R(\mathbf{r}_i - \mathbf{u}_i^{(1)}) = \frac{\partial}{\partial \mathbf{u}_i^{(1)}}\mathscr{E}_R. \tag{46}$$

In the process of enlargement, the effect of the forces acting at more distant contacts is taken into account. Passing from the scale R_1 to a scale $R_2 > R_1$, we obtain

$$\begin{aligned}\mathscr{E}_{R_2} &\equiv \mathscr{E}(\mathbf{r}_i - \mathbf{u}_i^{(2)} - \mathbf{w}_i^{(2)}) \\ &= \mathscr{E}_{R_1}(\mathbf{r}_i - \mathbf{u}_i^{(2)} - \mathbf{w}_i^{(2,1)}),\end{aligned} \tag{47}$$

where

$$\begin{aligned}\mathbf{w}_i^{(2,1)} &= \sum_{i\ni j}\hat{G}_{ij}\mathbf{f}_{R_1}(\mathbf{r}_i - \mathbf{u}_i^{(2)} - \mathbf{w}_i^{(2,1)}), \\ &R_1 \le |\mathbf{r}_1 - \mathbf{r}_2| \le R_2.\end{aligned} \tag{48}$$

If the difference between cut-offs R_2 and R_1 is very large, then the displacements $\mathbf{w}_i^{(2,1)}$ are small and expression (47) and (48) can be expanded in series in $\mathbf{w}_i^{(2,1)}$.

It is convenient to consider not the forces but their derivatives with respect to displacements,

$$f_R^{\alpha\beta} = \frac{\partial f^\alpha}{\partial u_\beta^{(1)}}, \quad f_R^{\alpha\beta\gamma} = \frac{\partial^2 f^\alpha}{\partial u_\beta^{(1)}\partial u_\gamma^{(1)}},$$

and so on. To see how the $f^{\alpha\beta\ldots}$ transform in passing from a scale R_1 to a scale R_2, it is necessary to differentiate expression (47) with respect to $\mathbf{u}_i^{(2)}$, after its expansion in $\mathbf{w}_i^{(2,1)}$. The displacements $\mathbf{w}_i^{(2,1)}$ themselves and their derivatives should be determined by Eq. (48). As a result, in the second order with respect to $\mathbf{w}$, we obtain

$$f_{R_2}^{\alpha\beta}(i) = \frac{\partial^2}{\partial u_\beta^{(1)}\partial u_\gamma^{(1)}} \times\left\{\mathscr{E}_{R_1}(i) - \sum_j G(ij)^{\gamma\delta}Q^{\gamma\delta}(ij)\right\}, \quad (49)$$

$$Q_{ij}^{\gamma\delta} = f_i^\gamma f_j^\delta - \sum_l G_{il}^{\mu\nu} f_i^{\gamma\mu} f_j^\delta - \sum_l G_{jl}^{\mu\nu} f_j^\gamma f_l^{\mu\delta},$$
$$f_i^\alpha = f_{R_1}^\alpha(i), \quad f_i^{\alpha\beta} = f_{R_1}^{\alpha\beta}(i). \quad (50)$$

The mean square $f^{\alpha\beta}$ is a quantity convenient for estimating the effects of mutual influence of the contacts. Because the mean value of an arbitrary quantity is zero, we obtain

$$\langle f_i^{\alpha\beta} f_j^{\gamma\delta}\rangle = -\langle f_i^{\alpha\beta\gamma} f_j^\delta\rangle = \langle f_i^{\alpha\beta\gamma\delta}\mathscr{E}_j\rangle$$
$$= \Gamma\delta_{ij}(\delta^{\alpha\beta}\delta^{\gamma\delta} + \delta^{\alpha\delta}\delta^{\beta\gamma} + \delta^{\alpha\gamma}\delta^{\beta\delta}). \quad (51)$$

To determine the variation of the effective charge Γ in passing from a scale R_1 to a scale R_2, expression (49) must be substituted in Eq. (51). Retaining the powers not exceeding four, taking into account that

$$G^{\alpha\beta}(r) = \frac{1}{8\pi E}\frac{1+\sigma}{1-\sigma}\left\{\frac{(3-4\sigma)\delta^{\alpha\beta}}{r} - \frac{r^\alpha r^\beta}{r^3}\right\}, \quad (52)$$

and performing the averaging, obtain the RG equation for effective charge Γ in the form

$$\Gamma_{R_2} = \Gamma_{R_1} + A\ln\frac{R_2}{R_1},$$
$$A = \pi\nu\left(\frac{1}{8\pi E}\frac{1+\sigma}{1-\sigma}\right)^2\{80(3-4\sigma)(1-\sigma)+9\}, \quad (53)$$

where ν is the number of contacts per unit area.

Renormalization group equation (53) is valid if $\Gamma\ln(R_2/R_1) \ll 1$. The exact expression for the effective charge Γ_R if Eq. (53) is differentiated with respect to $\ln R_2$,

$$\frac{d}{d\ln R}\Gamma_R = A\Gamma_R^2. \quad (54)$$

An initial condition for Eq. (54) can be obtained at $R = \rho$ (where ρ is the size of an individual contact):

$$\Gamma_\rho = \gamma = \frac{1}{8}\langle(\mathrm{div}\mathbf{f})^2\rangle. \quad (55)$$

The solution of RG equation (54) with initial condition (55) has the form

$$\Gamma_R = \frac{\gamma}{1 - A\gamma\ln(R/\rho)}. \quad (56)$$

Expression (56) is valid at $R < R_c$, where

$$R_c \approx \rho\exp\left\{-\frac{1}{\gamma A}\right\}. \quad (57)$$

Equation (57) solves the problem of the exponential in Eq. (16), if the dependence of the contact density ν on pressure is known. For the model of balls without a scatter in heights, we obtain

$$k_{\mathrm{fr}} \sim \exp\left\{-\frac{2^{17}(1-\sigma)}{3}R^2(16C_1C_2R^2)^{1/5}L\right\}, \quad (58)$$
$$L = [80(3-4\sigma)(1-\sigma)+9].$$

The result in this Appendix mainly repeats those in [8]. The difference in the derivation is that a similar results was obtain in [8] by summation of a perturbation series, while the RG procedure is used here.

COMMENTS, MARCH 2013

D. E. Khmelnitskii

Larkin and I were working on dry friction in Autumn 1978–Winter or 1979. From the very beginning of his work on pinning, Tolya saw the analogy with friction and spoke about this at numerous occasions. Finally, he suggested that I join him and study dry friction of two solid bodies. After the paper was written, we submitted it to JETP. Several days later, Tolya told me that Evgenii Michailovich Lifshits had spoken to him and asked to withdraw the paper:

I understand that you wrote a paper on physics.—E.M. said—But if JETP publishes it, we will be flooded by articles written by engineers.

So, Tolya, took the paper from the Editorial office.[3] At about that time (March 1979), the text was translated by the staff translator at the Landau Institute and printed out as a Landau Institute preprint. A bit later, Tolya suggested to submit the

[3] Seven or eight years later, after JETP E. M. had passed away and I was appointed Deputy Editor at JETP, Tolya asked me with a caustic smile whether I would reject the paper on dry friction if it was submitted at that time.

English text to Physical Review A. We submitted and received a report, which, as I understand now, was pretty neutral on the subject matter and mentioned our poor English. Still, it sounded a rejection to us. We were then involved in a very exciting work with Lev Gorkov on weak localization, and the paper on dry friction was left behind.

Since the preprint was published and members of the Landau Institute have spread around the globe, this work was not completely forgotten. A number of colleagues requested the preprint from me and it has been cited in publications about dry friction.[4] Now, 34 years after it was written, this article can be available to the broad readership.[5]

[4] Most notably, in the paper by C. Caroli and P. Nozieres, *Hysteresis and elastic interactions of micro–asperities in dry friction*, European Physical Journal B **4**, 233–246 (1998).

[5] This text is largely based on the English translation made in 1979 by L. I. Velyuts, with a small number of recent corrections.

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