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The Mobility Edge Problem: Continuous Symmetry and a Conjecture

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An apparently overlooked symmetry of the disordered electron problem is derived. It yields the well-known Ward-identity connecting the one- and two-particle Green's function. This symmetry and the apparent shortrange behaviour of the averaged one-particle Green's function are used to conjecture that the critical behaviour near the mobility edge coincides with that of interacting matrices which have two different eigenvalues of multiplicity zero (due to replicas). As a consequence the exponent s of the d.c. conductivity is expected to approach 1 for real matrices and 1/2 for complex matrices as the dimensionality of the system approaches two from above. In two dimensions no metallic conductivity is expected.

1. Introduction

Symmetries play a crucial rôle in critical phenomena. They are important to determine the universality class and thus the critical behaviour of a system. Apparently an internal continuous symmetry of the disordered electron problem has gone unnoticed. It will be derived from the field theoretic formulation [1, 2] in Sect. 2. The energy difference $\omega = z_1 - z_2$ acts as the symmetry breaking source and the difference of the Green's functions with energies above and below the real axis plays the rôle of an order parameter. The Ward-identity [3] which connects the zero-momentum two-particle Green's function to the one-particle Green's function is obtained from this symmetry in the same manner as that between the transverse susceptibility and the magnetization of an isotropic ferromagnet.

This symmetry as well as the apparent short-range behaviour of the averaged one-particle Green's function are used to conjecture that the mobility edge behaviour coincides with the critical behaviour of interacting matrices (Sect. 3). These matrices have two different eigenvalues corresponding to the averaged Green's functions with multiplicities zero due to the vanishing number of replicas.

On the basis of this conjecture we deduce from renormalization-group calculations by Brézin, Hikami, and Zinn-Justin [4] that in the limit $d \to 2+$ the exponent s for the d.c. conductivity $\sigma_{\mathrm{d.c.}} \sim (E - E_a)^s$ approaches 1 for the real matrix ensemble and 1/2 for the phase invariant ensemble [5]. The result for the real matrix ensemble agrees with that obtained by Oppermann and the present author [5] on the basis of a $1/n$ expansion for a system with n orbitals per site. The "order parameter", that is, the density of states need not vanish at the mobility edge as expected. The analogies between a disordered electronic system and a ferromagnet with continuous broken symmetry observed in [5] become obvious here.

2. Symmetry

Consider the one-particle tight-binding Hamiltonian

$$H = \sum_{rr'} v_{rr'} |r\rangle \langle r'| \tag{2.1}$$

for the systems of an ensemble with a given probability distribution of the v's. Then similarly to Nitzan, Freed and Cohen [1] and to Aharony and Imry [2] we express the averaged one-particle Green's function

$$G(r, r', z_p) = \overline{\langle r| (z_p - H)^{-1} |r'\rangle} \tag{2.2}$$

and the two-particle Green's function

$$K(r_1, r_1', z_1, r_2, r_2', z_2) = \overline{\langle r_1| (z_1 - H)^{-1} |r_1'\rangle \langle r_2| (z_2 - H)^{-1} |r_2'\rangle} \tag{2.3}$$

as expectation values

$$G(r, r', z_p) = \langle S_a^{p*}(r)\, S_a^p(r')\rangle \tag{2.4}$$

$$K(r_1, r_1', z_1, r_2, r_2', z_2) = \langle S_a^{1*}(r_1)\, S_a^1(r_1')\, S_b^{2*}(r_2)\, S_b^2(r_2')\rangle \tag{2.5}$$

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of a system with Hamiltonian

$$\mathscr{H}=\beta\sum_p z_p \sum_{ra} S_a^{p*}(r)\, S_a^p(r)+\mathscr{H}_0(\hat{Q}), \tag{2.6}$$

$$\exp(-\mathscr{H}_0(\hat{Q}))=\overline{\exp(\beta\sum_{rr'} v_{rr'}\hat{Q}_{rr'})} \tag{2.7}$$

$$\hat{Q}_{rr'}=\sum_{pa} S_a^p(r)\, S_a^{p*}(r') \tag{2.8}$$

in the limit of $m_p \to 0$ components S_a^p. The bar indicates the average over the ensemble. The brackets $\langle\ldots\rangle$ indicate the average over the fields S,

$$\langle A\rangle = Z^{-1}\prod_{par}(\int d\,S_a^p(r))A\exp(-\mathscr{H}) \tag{2.9}$$

$$Z=\prod_{par}(\int d\,S_a^p(r))\exp(-\mathscr{H}). \tag{2.10}$$

We have chosen phase factors different from those in [1, 2]. For real (symmetric) matrices v, S^* stands for

$$S_a^{p*}(r)=S_a^p(r) \tag{2.11}$$

and the path of integration in (2.9, 10) is chosen so that $S_a^p(r)$ equals $(1-i\,\mathrm{sign}\,\mathrm{Im}\,z_p)/\sqrt{2}$ times a real number. Thus S^* has a meaning different from the usual one. For complex (hermitian) matrices v, we decompose

$$S_a^p(r)=S_a^{p'}(r)+i\,S_a^{p''}(r)$$
$$S_a^{p*}(r)=S_a^{p'}(r)-i\,S_a^{p''}(r) \tag{2.12}$$

and again choose S' and S'' to be $(1-i\,\mathrm{sign}\,\mathrm{Im}\,z_p)/\sqrt{2}$ times real numbers and

$$\int dS=\int dS'\int dS''. \tag{2.13}$$

The constant β equals 1/2 and 1 for real and complex matrices, resp.
Now replace (2.6) by the more general expression

$$\mathscr{H}=\sum_{pp'aa'rr'} h_{aa'}^{pp'}(r,r')\, S_a^{p*}(r)\, S_{a'}^{p'}(r')+\mathscr{H}_0(\hat{Q}) \tag{2.14}$$

Then (2.6) is recovered for

$$h_{aa'}^{pp'}(r,r')=\beta z_p \delta_{pp'}\delta_{aa'}\delta_{rr'}. \tag{2.15}$$

A variation of the sources h yields the expectation values as derivatives of the "free energy"

$$F=\ln Z$$

$$G(r,r',z_p)=-\partial F/\partial h_{aa}^{pp}(r,r') \tag{2.16}$$

$$K(r_1,r_1',z_1,r_2,r_2',z_2)=\frac{\partial^2 F}{\partial h_{ab}^{12}(r_2,r_1')\,\partial h_{ba}^{21}(r_1,r_2')}$$
$$=\frac{\partial^2 F}{\partial h_{aa}^{11}(r_1,r_1')\,\partial h_{bb}^{22}(r_2,r_2')}$$
$$+G(r_1,r_1',z_1)\,G(r_2,r_2',z_2). \tag{2.17}$$

The Hamiltonian $\mathscr{H}_0(\hat{Q})$ as well as $\mathscr{H}$ given by (2.6) are invariant under orthogonal and unitary global transformations of the groups $O(m_1)\times O(m_2)$ and $U(m_1)\times U(m_2)$, resp. in the space of replicas. However, $\mathscr{H}(\hat{Q})$ is invariant under the larger group $O(m_1+m_2)$ and $U(m_1+m_2)$ of transformations,

$$S_a^p(r)=\sum_{p'a'} T_{aa'}^{pp'}\,\tilde{S}_{a'}^{p'}(r). \tag{2.18}$$

The sources h transform like

$$\tilde{h}_{aa'}^{pp'}(r,r')=\sum_{qq'bb'} T_{ab}^{pq*}\, T_{a'b'}^{p'q'}\, h_{bb'}^{qq'}(r,r'). \tag{2.19}$$

Thus one has

$$F(\tilde{h})=F(h), \tag{2.20}$$

provided the transformation is so close to the unit transformation that the partition function Z, (2.10) still converges. For (2.6) the convergency depends on the imaginary part of z_1 and z_2. The limit $z_{1,2}=E\pm i0$, E real, is the limit of convergency and simultaneously the limit in which $\mathscr{H}$ has the full continuous $O(m_1+m_2)$ and $U(m_1+m_2)$ symmetry, resp. For $z_1-z_2\neq 0$ the symmetry is broken by the first term on the right hand side of (2.6). Thus z_1-z_2 is the symmetry breaking source.
From the symmetry (2.20) the Ward-identity [3]

$$G(r_1,r_1',z_1)-G(r_1,r_1',z_2)$$
$$=(z_2-z_1)\sum_r K(r_1,r,z_1,r,r_1',z_2) \tag{2.21}$$

can be derived. The orthogonal transformation

$$\tilde{h}_{aa'}^{pp'}=h_{aa'}^{pp'}+\omega\delta_{a1}(\delta_{p1}h_{1a'}^{2p'}-\delta_{p2}h_{1a'}^{1p'})$$
$$+\omega\delta_{a'1}(\delta_{p'1}h_{a1}^{p2}-\delta_{p'2}h_{a1}^{p1}), \tag{2.22}$$

ω real and infinitesimal, yields

$$\sum_{p'a'rr'}(h_{1a'}^{2p'}\,\partial F/\partial h_{1a'}^{1p'}-h_{1a'}^{1p'}\,\partial F/\partial h_{1a'}^{2p'})$$
$$+\sum_{parr'}(h_{a1}^{p2}\,\partial F/\partial h_{a1}^{p1}-h_{a1}^{p1}\,\partial F/\partial h_{a1}^{p2})=0 \tag{2.23}$$

where all sources h carry the arguments (r,r'). Take the derivative of (2.23) with respect to $h_{11}^{12}(r_1,r_1')$ and evaluate it for h given by (2.15)

$$-\partial F/\partial h_{11}^{22}(r_1,r_1')+\partial F/\partial h_{11}^{11}(r_1,r_1')$$
$$+\beta\,\partial/\partial h_{11}^{12}(r_1,r_1')\sum_r\{z_2[\partial F/\partial h_{11}^{12}(r,r)+\partial F/\partial h_{11}^{21}(r,r)]$$
$$-z_1[\partial F/\partial h_{11}^{21}(r,r)+\partial F/\partial h_{11}^{12}(r,r)]\}=0. \tag{2.24}$$

For real matrices v one has $\partial^2 F/\partial h^{12}\partial h^{12}=\partial^2 F/\partial h^{12}\partial h^{21}$, whereas for complex matrices v, $\partial^2 F/\partial h^{12}\partial h^{12}=0$ holds. Thus with (2.16) and (2.17) one obtains the Ward identity (2.21).

3. Matrix Model and Mobility Edge Behaviour

The quantities of interest are the expectation values of products of the bilinear forms $S^* S$. This suggests to introduce the composite variables

$$Q_{aa'}^{pp'}(r, r') = S_{a'}^{p'*}(r')\, S_a^p(r). \tag{3.1}$$

Such variables were used by Aharony and Imry [2] and by Schuster [6]. These authors, however, introduced Q only for p different from p'. Here we also keep Q for $p=p'$ explicitly to preserve the full symmetry. One expects that $G(r, r', z_p) = \langle Q_{aa}^{pp}(r', r)\rangle$ decays rapidly as $|r-r'|$ tends to infinity. On a length scale large in comparison to this phase correlation length it should be possible to express the problem in terms of the local field

$$Q_{aa'}^{pp'}(r) = Q_{aa'}^{pp'}(r, r) \tag{3.2}$$

only, as in [2] and [6].

Let us consider the eigenvalues of the matrix Q. In the symmetry breaking interaction (2.6) one has

$$\langle Q_{aa'}^{pp'}(r)\rangle = \delta_{pp'}\,\delta_{aa'}\, G(z_p) \tag{3.3}$$

where $G(z_p)$ is the diagonal term $G(r, r, z_p)$. This suggests that the main contribution comes from matrices with m_1 eigenvalues close to $G(z_1)$ and m_2 eigenvalues close to $G(z_2)$. Let us assume that fluctuations in the eigenvalues of $Q(r)$ are irrelevant similarly as the fluctuations of the length of the vector in the n-vector-model. Then we may assume that Q has m_1 eigenvalues λ_1 and m_2 eigenvalues λ_2. Any local potential with the full symmetry $O(m_1+m_2)$ and $U(m_1+m_2)$, resp. is a function of λ_1 and λ_2 only and thus a constant. The most simple interaction for Q defined in the continuous real space is

$$\mathscr{H}_0 = \tfrac{1}{2}\beta \hat{K} \int d^d r \,\mathrm{tr}\,(\nabla Q(r)\, \nabla Q(r)) \tag{3.4}$$

with some coupling constant $\hat{K}$. There are no other interactions proportional ∇^2 with the desired symmetry. We conjecture that the critical behaviour at the mobility edge can be described by the Hamiltonian (3.4). A symmetry breaking term similar to (2.6) has to be added

$$\mathscr{H} = \beta\, v^{-1} \sum_{pa} z_p \int d^d r\, Q_{aa}^{pp}(r) + \mathscr{H}_0. \tag{3.5}$$

where v is the volume per lattice site r. The matrices Q are given by

$$Q = T \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} T^{+} \tag{3.6}$$

where the group of transformations T depends on the symmetry of the interactions (2.1) of the ensemble. Here we have given two possibilities, orthogonal transformations for real matrices and unitary matrices for complex matrices. The physical relevance of other groups has to be investigated.

Since Q has the eigenvalues λ_1 and λ_2, only, it obeys

$$(Q-\lambda_1)(Q-\lambda_2) = 0 \tag{3.7}$$

which yields for small Q^{12}, Q^{21} and $Q^{11} \approx \lambda_1$

$$Q^{11} = \tfrac{1}{2}(\lambda_1+\lambda_2) + [(\tfrac{1}{2}(\lambda_1-\lambda_2))^2 - Q^{12} Q^{21}]^{\frac{1}{2}} \tag{3.8}$$

and similarly for Q^{22}. An expansion in powers of $Q^{12} Q^{21}$ yields

$$\langle Q_{aa}^{11}\rangle = \lambda_1 \delta_{ab} - (\lambda_1-\lambda_2)^{-1} \sum_c \langle Q_{ac}^{12} Q_{cb}^{21}\rangle + \dots \tag{3.9}$$

Since the summation runs over $m_2=0$ indices c one obtains exactly

$$\langle Q_{ab}^{11}\rangle = \lambda_1 \delta_{ab}. \tag{3.10}$$

Thus the eigenvalues λ_p are determined by the one-particle Green's function

$$\lambda_p = G(z_p). \tag{3.11}$$

This shows that for smooth λ as a function of energy there is no critical behaviour of G at the mobility edge as expected. Similarly the two-particle Green's function $K(z_1, z_1) = \langle Q^{11} Q^{11}\rangle$ does not show any critical behaviour.

For small elongations of the "transverse" components Q^{12} and Q^{21} the "longitudinal" components Q^{11} and Q^{22} can be expressed in terms of the Q^{12} and Q^{21} according to (3.8). Then $\mathscr{H}_0$ can be expressed in terms of the transverse components only. The situation is similar to the nonlinear σ-model where the longitudinal component can be expressed in terms of the transverse components. In both cases a term for the invariant measure appears after elimination of the longitudinal components which is independent of the coupling $\hat{K}$.

The conjecture is supported by observations made on the local gauge invariant models with n orbitals per site for which a $1/n$ expansion has been derived diagrammatically in [5]. For this model a Lagrangian L can be constructed which generates the diagrams of [5] (this and the following statements on this model will be published elsewhere). The fields of this Lagrangian are matrices $Q(r)$ with components $Q_{aa'}^{pp'}(r)$ but without the restriction (3.7). After elimination of the "massive" components Q^{pp} one obtains a Lagrangian $\mathscr{L}$ with contributions of order n, n^0, n^{-1}, etc. The leading contribution (of order n) vanishes for small momenta of the Q like q^2. The contributions of order q^2 have exactly the form (3.4) with Q^{pp} replaced by (3.8) and

$$\hat{K}(\lambda_1-\lambda_2)^2 = -n R^2 a^{-2}(1 - E^2 E_0^{-2}) \tag{3.12}$$

(for notation on the right-hand side of this equation see [5]) where the over-all minus sign is due to the imaginary difference $\lambda_1-\lambda_2$.

Introducing a "temperature" t by

$$(2^{d-1}\pi^{d/2}\Gamma(d/2)\, t)^{-1} = \hat{K}(\lambda_1-\lambda_2)^2 \tag{3.13}$$

the estimate [5] for the mobility edge of the real matrix ensemble yields

$$t_c = -(d-2)/2 + O(d-2)^2. \tag{3.14}$$

This is in agreement with the non-trivial zero of the W-function calculated by Brézin, Hikami, and Zinn-Justin [4] for the model (3.4) with orthogonal T,

$$W(t) = (d-2)\,t + 2t^2 + O(t^4), \tag{3.15}$$

who obtain $t_c = -(d-2)/2 + O(d-2)^3$.
According to [7] the exponent s of the d.c. conductivity is given by

$$s = (d-2)\,\nu = (d-2)/y \tag{3.16}$$

where y is the critical exponent of the relevant perturbation

$$y = -W'(t_c) \tag{3.17}$$

which yields

$$s = 1 + O(d-2)^2 \tag{3.18}$$

again in agreement with the estimate $s = 1 + O(d-2)$ in [5].
For unitary T Brézin, Hikami, and Zinn-Justin [4] obtain

$$W(t) = (d-2)\,t - 2t^3 + O(t^4). \tag{3.19}$$

Thus for the phase invariant ensemble one expects the mobility edge for

$$t_c = -(d/2-1)^{\frac{1}{2}} + O(d-2) \tag{3.20}$$

and

$$s = \tfrac{1}{2} + O(d-2)^{\frac{1}{2}} \tag{3.21}$$

slightly above $d=2$ dimensions. The Lagrangian $\mathscr{L}(Q^{12})$ mentioned above shows the symmetry properties as a function of the fields h.

The stability of these fixed points needs further investigation. Since $t_c = 0$ for $d=2$ no metallic conductivity for these systems is expected in two dimensions in agreement with Götze, Prelovšek and Wölfle [8] and Abrahams, Anderson, Licciardello and Ramakrishnan [9].

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