General Theory of Spin-Wave Interactions*

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An ideal model of a ferromagnet is studied, consisting of a lattice of identical spins with cubic symmetry and with isotropic exchange coupling between nearest neighbors. The aim is to obtain a complete description of the thermodynamic properties of the system at low temperatures, far below the Curie point. In this temperature region the natural description of the states of the system is in terms of Bloch spin waves. The nonorthogonality of spin-wave states raises basic difficulties which are examined and overcome.

The following new results are obtained: a practical method for calculating thermodynamic quantities in terms of a nonorthogonal set of basic states; a proof that in 3 dimensions there do not exist states (shown by Bethe to exist in a one-dimensional chain of spins) in which two spins are bound together into a stable complex and travel together through the lattice; a calculation of the scattering cross section of two spin waves, giving a mean free path for spin-spin collisions proportional to $T^{-7/2}$ at low temperatures; and an exact formula for the free energy of the system, showing explicitly the effects of spin-wave interactions.

Quantitative results based on this theory will be published in a second paper.

1. INTRODUCTION

BLOCH[1] invented the concept of a spin wave, which consists of a single reversed spin distributed coherently over a large number of otherwise aligned atomic spins in a crystal lattice. He showed first that the low-energy excited states of a ferromagnet would be of this character, and second that the resulting thermodynamic properties of a ferromagnet at low temperature were in agreement with experiment.

The theory of Bloch explicitly assumes that the density of reversed spins is so small that the effects of obstruction and interaction between two or more spin waves may be neglected. This is an approximation which will certainly be good at sufficiently low temperatures, less good at higher temperatures. When spin-wave interactions are neglected, the whole theory becomes linear in the spin-wave amplitudes. Herring and Kittel[2] took advantage of the linearity to construct a very simple phenomenological spin-wave theory which is independent of the underlying atomic structure of the ferromagnet. But the question, how good such a linear approximation should be, remained open.

Bethe[3] made a thorough study of the effects of spin-wave interactions in a one-dimensional chain of spins (in which case there is no ferromagnetism). He showed that in addition to the elementary Bloch spin waves there exist excitations in which a block of two or more reversed spins travel together through the chain. But it did not seem easy to apply Bethe's methods to the 3-dimensional case. The extension of Bethe's results to 3 dimensions is made for the first time in Sec. 6 of the present paper.[4]

A new attack on the problem of spin-wave interactions was opened by Holstein and Primakoff.[5] They considered the behavior of a 3-dimensional ferromagnetic array of spins in an external magnetic field. They succeeded in defining a set of coordinates which describe accurately the quantum state of the system and which have the appearance of "spin-wave amplitudes." In terms of these coordinates the Hamiltonian of the system splits into two parts, one quadratic in the amplitudes and one of higher order. The quadratic part alone would give a theory of noninteracting spin-waves, identical with the linear approximation of Bloch. The nonquadratic part represents the effects of interaction between spin waves. It therefore appeared that a consistent treatment of spin-wave interactions would be possible, taking the quadratic part of the Hamiltonian as a first approximation and dealing with the nonquadratic part by perturbation theory. However, it turns out that this is only possible when the external field is strong. The nonquadratic part of the Hamiltonian is large, and for low-frequency spin waves in a weak external field the nonquadratic part dominates the quadratic part. The Holstein-Primakoff formalism is thus essentially nonlinear and unamenable to exact calculations. In fact, we shall see in Sec. 3 that the interaction between low-frequency spin waves is quite weak and is grossly overestimated by the Holstein-Primakoff Hamiltonian. The reason for this failure is that the Holstein-Primakoff coordinates are an artificial creation and do not answer the physical requirements of spin-wave amplitudes.

The program of the present paper is to develop a mathematical apparatus for calculating the effect of spin-wave interactions systematically and to high precision. The calculation of these effects will be carried through in detail in a subsequent paper. We shall confine our attention to a highly idealized model of a

* Research supported by the National Science Foundation.
[1] F. Bloch, Z. Physik **61**, 206 (1930); **74**, 295 (1932).
[2] C. Herring and C. Kittel, Phys. Rev. **81**, 869 (1951).
[3] H. A. Bethe, Z. Physik **71**, 205 (1931); A. Sommerfeld and H. A. Bethe, *Handbuch der Physik* (Verlag Julius Springer, Berlin, 1933), Vol. 24, Part 2, 604–618.
[4] We extend to 3 dimensions only the part of Bethe's work referring to spin complexes with 2 reversed spins. In particular, we do not claim to have extended Bethe's main result, which is the exact determination of the ground-state wave function of a one-dimensional antiferromagnet.
[5] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

ferromagnet, namely a 3-dimensional array of fixed spins with cubic symmetry and purely isotropic exchange coupling of nearest neighbors. The aim is to understand accurately the behavior of this simple model, so that the deviations of real ferromagnets from such behavior can be attributed to physical effects which are here neglected.

Practical interest in the question of spin-wave interactions arose from a remark of Néel.[6] He pointed out that the customary interpretation of the low-temperature spontaneous magnetization of ferromagnets rested on the linearized Bloch theory, and that the two published methods of calculation[7] of the correction to the magnetization produced by spin-wave interactions were in disagreement. The method of the present paper settled the disagreement by showing that both calculations were wrong. It turns out that the effect of spin-wave interactions is smaller than either of the previous calculations envisaged, and therefore the customary use of the linear theory is justified.

The analysis of the spin-wave phenomenon presented in this paper goes very far beyond what would conceivably be required in practical applications. The purpose of the detailed analysis is to reach an understanding of the mathematical complexity inherent in even such a simple model as we are considering. In the process of understanding the model, we have developed some new mathematical tricks and tools which may have wider application. In this respect our work is comparable to what has been done with the Ising model,[8] though we may justifiably claim that our model has considerably greater similarity than the Ising model has to a real ferromagnet.

The analysis is directed toward a calculation of the free energy A of the system, as a series expansion in ascending powers of the absolute temperature T. The expansion parameter is the dimensionless ratio (kT/J), where k is Boltzmann's constant and J is the exchange energy between two neighboring spins. Once the free energy is calculated, most of the other physical quantities of interest can be easily deduced. Through the analysis there runs a certain analogy to the situation in quantum electrodynamics, where the classical problem[9] was the calculation of the scattering matrix S as a power series in (e^2/hc). In both cases, the power series developments of S or A can be carried to any desired order, and yet the quantities S and A are not in principle determined.

In the analysis of the ferromagnet, we make an expansion in powers of (kT/J), and explicitly neglect effects proportional to $\exp[-J/kT]$ which tend to zero at low temperatures faster than any power of T. But it is clear that effects of order $\exp[-J/kT]$ play a decisive role at temperatures around the order-disorder transition point where kT and J are comparable. Therefore the low-temperature power-series development cannot determine the behavior of the ferromagnet in the transition region, even if all terms in the expansion are supposed accurately known. The series is an asymptotic expansion for the free energy, but does not determine the free energy. If the series happens to be convergent, its sum need still not be exactly equal to the free energy [as a trivial example of this, consider the Ising model where the entire free energy goes to zero like $\exp(-J/kT)$ at low temperatures and so all coefficients in the power series are zero, but the free energy is not zero].

It is interesting to speculate that the ideal ferromagnet is a model of the kind of behavior to be expected in any nonlinear field theory such as quantum electrodynamics or meson theory. We have seen that an expansion in powers of $[kT/J]$ gives no hint of the behavior of the ferromagnet at the Curie point. It is likely that weak-coupling treatments of field theory, in which $(e^2/\hbar c)$ is treated as small, give no hint of what really happens in the domain of strong coupling. The very concept of a spin wave seems to be limited to the domain of low temperatures where the power-series expansion is meaningful, and it may be that the concepts of contemporary field theory are likewise meaningful only within the domain of weak-coupling approximations.

2. DEFINITION OF SPIN WAVES

The model to be discussed is a finite cubic crystal with periodic boundary conditions. Each atom is labeled by a lattice vector $\mathbf{j}$, and there is a fixed integer n such that the atom $(\mathbf{j}+n\mathbf{k})$ is identical with the atom $\mathbf{j}$ for any two lattice vectors $\mathbf{j}$, $\mathbf{k}$. The total number of atoms is $N=n^3$, and the volume of the crystal is NV, where V is the volume of a unit cell.

To each atom $\mathbf{j}$ is attached a spin vector $\mathbf{S}_j$ obeying the usual commutation rules, which it is convenient to write in the form

$$\begin{aligned}[S_j{}^z,S_k{}^+]&=\delta_{jk}S_j{}^+,\\ [S_j{}^z,S_k{}^-]&=-\delta_{jk}S_j{}^-,\\ [S_j{}^+,S_k{}^-]&=2\delta_{jk}S_j{}^z,\end{aligned} \tag{1}$$

with

$$S_j{}^{\pm}=S_j{}^x\pm iS_j{}^y. \tag{2}$$

Each spin has magnitude S; this implies the operator identity

$$S_j{}^-S_j{}^+=(S-S_j{}^z)(S+S_j{}^z+1). \tag{3}$$

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[8] G. F. Newell and E. W. Montroll, Revs. Modern Phys. **25**, 353 (1953). This article gives a full bibliography of earlier work on the subject.

[9] F. J. Dyson, Phys. Rev. **75**, 486 and 1736 (1949).

The dynamical properties of the model are expressed in the Hamiltonian

$$\mathcal{H}=\mathcal{E}+\mathfrak{M}, \tag{4}$$

$$\mathcal{E}=-\tfrac{1}{2}J\sum_{j\delta}\mathbf{S}_j\cdot\mathbf{S}_{j+\delta}, \tag{5}$$

$$\mathfrak{M}=L\sum_j S_j^z. \tag{6}$$

The summation in (5) extends over all lattice vectors $\mathbf{j}$ and over the vectors $\boldsymbol{\delta}$ which join an atom to one of its nearest neighbors. Thus $\mathcal{E}$ is an isotropic exchange coupling of magnitude J between nearest neighbors, and we shall suppose J positive so that the system is ferromagnetic at low temperatures. The magnetic energy $\mathfrak{M}$ represents the effect of a uniform external magnetic field, directed along the negative z-axis, of magnitude

$$H=SL/m, \tag{7}$$

where m is the magnetic moment of each spin. Magnetic dipole coupling between the spins is not included in this model.

The reciprocal lattice of the crystal is here defined as the set of vectors $\boldsymbol{\lambda}$ such that for all lattice-vectors $\mathbf{j}$ the quantities

$$(n/2\pi)(\boldsymbol{\lambda}\cdot\mathbf{j}) \tag{8}$$

are integers. Reciprocal lattice vectors are denoted by Greek letters $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\rho}$, $\boldsymbol{\sigma}$, and $(\boldsymbol{\lambda}+n\boldsymbol{\mu})$ is identified with $\boldsymbol{\lambda}$. There are N reciprocal lattice vectors, and the reciprocal lattice cell volume is

$$v=(2\pi)^3(NV)^{-1}. \tag{9}$$

The spin operators attached to the reciprocal lattice are defined by

$$\mathbf{S}_\lambda=N^{-\frac{1}{2}}\sum_j\exp(i\boldsymbol{\lambda}\cdot\mathbf{j})\mathbf{S}_j, \tag{10}$$

and their commutation rules follow from (1):

$$\begin{aligned}&[S_\lambda^z,S_\mu^+]=N^{-\frac{1}{2}}S_{\lambda+\mu}^+,\\ &[S_\lambda^z,S_\mu^-]=-N^{-\frac{1}{2}}S_{\lambda+\mu}^-,\\ &[S_\lambda^+,S_\mu^-]=2N^{-\frac{1}{2}}S_{\lambda+\mu}^z.\end{aligned} \tag{11}$$

The ground state of the whole system is the state $|0\rangle$ defined by

$$S_j^-|0\rangle=0,\quad S_j^z|0\rangle=-S|0\rangle \tag{12}$$

for all $\mathbf{j}$, or equivalently

$$S_\lambda^-|0\rangle=0,\quad S_\lambda^z|0\rangle=-N^{\frac{1}{2}}S\delta_{\lambda 0}|0\rangle \tag{13}$$

for all $\boldsymbol{\lambda}$. Following Bloch[1] we define a spin-wave state $|1_\lambda\rangle$, containing a single spin wave with wave vector $\boldsymbol{\lambda}$, by

$$|1_\lambda\rangle=(2S)^{-\frac{1}{2}}S_\lambda^+|0\rangle. \tag{14}$$

These states are properly normalized and are orthogonal to each other and to $|0\rangle$. The problem now arises, how to define spin-wave states with more than one spin-wave. We shall follow Bloch and use the natural generalization of (14). Let (a) represent any set of non-negative integers a_λ, one attached to each reciprocal lattice vector $\boldsymbol{\lambda}$. Then the spin-wave state $|a\rangle$, containing a_λ spin waves with wave vector $\boldsymbol{\lambda}$, is defined by

$$|a\rangle=\prod_\lambda[(2S)^{-\frac{1}{2}a_\lambda}(a_\lambda!)^{-\frac{1}{2}}(S_\lambda^+)^{a_\lambda}]|0\rangle. \tag{15}$$

The order of factors in the product is immaterial since they all commute.

As soon as $\sum a_\lambda>1$, the states (15) are neither normalized nor orthogonal to each other. From this fact arise all the conceptual difficulties in thinking about spin waves. The states (15) are much more numerous than the total number $(2S+1)^N$ of independent states which the system possesses. So the variables (a) are a highly redundant set of variables for describing the system. Nevertheless we shall find it convenient to use these variables, and to take their redundance into account explicitly. It happens that at low temperatures the only important states $|a\rangle$ are those with $\sum a_\lambda$ small compared with N, and these states are approximately orthogonal to each other. They come closer to orthogonality the lower the temperature. Thus the idea of a spin wave, as it is here defined by Eq. (15), is an idea which has a perfectly sharp meaning only at zero temperature and loses its meaning entirely as the temperature rises to the Curie point.

3. KINEMATICAL AND DYNAMICAL SPIN-WAVE INTERACTIONS

The nonorthogonality of the states (15) produces an interaction between spin waves which we call the kinematical interaction. The physical cause of this interaction is the fact that more than $(2S)$ units of reversed spin cannot be attached to the same atom simultaneously. There is therefore a certain statistical hindrance to any dense packing of many spin waves within a limited volume.

There is another spin-wave interaction which arises from the fact that the Hamiltonian (4) is not diagonal in the states (15). This we call the dynamical interaction. To obtain this interaction explicitly, we shall calculate the effect of the Hamiltonian operating on (15). Using (10), the Hamiltonians (5) and (6) may be written

$$\mathcal{E}=-\tfrac{1}{2}J\sum_\lambda\gamma_\lambda\mathbf{S}_\lambda\cdot\mathbf{S}_{-\lambda}, \tag{16}$$

$$\gamma_\lambda=\sum_\delta\exp(i\boldsymbol{\delta}\cdot\boldsymbol{\lambda}), \tag{17}$$

$$\mathfrak{M}=LN^{\frac{1}{2}}S_0^z. \tag{18}$$

To calculate $\mathcal{H}|a\rangle$, we commute the operator $\mathcal{H}$ through the operators S_λ^+ appearing in Eq. (15) until $\mathcal{H}$ operates directly on the ground state $|0\rangle$. This gives a sum of terms involving the commutators

$$\begin{aligned}Q_\rho=[\mathcal{H},S_\rho^+]&=LS_\rho^+\\ &-\tfrac{1}{2}JN^{-\frac{1}{2}}\sum_\lambda(\gamma_\lambda-\gamma_{\lambda-\rho})(S_\lambda^zS_{\rho-\lambda}^+-S_\lambda^+S_{\rho-\lambda}^z),\end{aligned} \tag{20}$$

plus a term $E_0|a\rangle$ with E_0 given by

$$\mathcal{H}|0\rangle=E_0|0\rangle,\quad E_0=-\tfrac{1}{2}JNS^2\gamma_0-LNS. \tag{21}$$

Next we commute the Q_ρ again through the operators S_λ^+ until Q_ρ operates directly on $|0\rangle$. This gives a sum of terms involving the commutators

$$R_{\rho\sigma}=[Q_\rho,S_\sigma^+]=-\tfrac{1}{2}JN^{-1}\sum_\lambda \Gamma_{\rho\sigma}{}^\lambda S_{\sigma+\lambda}^+S_{\rho-\lambda}^+, \quad (22)$$

$$\Gamma_{\rho\sigma}{}^\lambda=\gamma_\lambda-\gamma_{\lambda-\rho}-\gamma_{\lambda+\sigma}+\gamma_{\lambda+\sigma-\rho}, \quad (23)$$

plus a sum of terms $(L+\epsilon_\rho)|a\rangle$ with ϵ_ρ given by

$$Q_\rho|0\rangle=(L+\epsilon_\rho)S_\rho^+|0\rangle, \quad \epsilon_\rho=JS(\gamma_0-\gamma_\rho). \quad (24)$$

The process of commutation comes to an end after two steps because

$$[R_{\rho\sigma},S_\lambda^+]=0. \quad (25)$$

The result of this process is a formula

$$\mathfrak{H}|a\rangle=[E_0+\textstyle\sum_\lambda a_\lambda(L+\epsilon_\lambda)]|a\rangle+\sum_b Q_{ab}|b\rangle, \quad (26)$$

where the second sum extends over spin-wave states $|b\rangle$ which are obtained from $|a\rangle$ by replacing one pair of spin waves $(\boldsymbol{\varrho},\boldsymbol{\sigma})$ by a pair $(\boldsymbol{\varrho}-\boldsymbol{\lambda}, \boldsymbol{\sigma}+\boldsymbol{\lambda})$. The Q_{ba} are numerical coefficients containing the $\Gamma_{\rho\sigma}{}^\lambda$ defined by Eq. (23). In Eq. (26) the Hamiltonian is explicitly separated into a diagonal part representing the energy of free spin waves, and a nondiagonal part representing a simple scattering of one spin wave by another. The nondiagonal part of Eq. (26) is the dynamical spin-wave interaction.

The main reason for using the spin-wave states defined by Eq. (15) is that the resulting dynamical spin-wave interaction is very weak. According to Eqs. (17) and (23),

$$\Gamma_{\rho\sigma}{}^\lambda=\textstyle\sum_\delta \exp(i\boldsymbol{\delta}\cdot\boldsymbol{\lambda})[1-\exp(-i\boldsymbol{\delta}\cdot\boldsymbol{\varrho})]\times[1-\exp(i\boldsymbol{\delta}\cdot\boldsymbol{\sigma})]. \quad (27)$$

These coefficients become small when the spin waves $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$ have wavelengths long compared with $\boldsymbol{\delta}$, irrespective of the value of $\boldsymbol{\lambda}$. Therefore for a low-temperature spin-wave state $|a\rangle$ containing only long-wavelength spin waves, all interaction matrix elements Q_{ba} are uniformly small, and the Hamiltonian $\mathfrak{H}$ is almost diagonal. The effects of the interaction are reduced still further when (27) is averaged over the directions of $\boldsymbol{\varrho}$ and $\boldsymbol{\sigma}$.

The Holstein-Primakoff spin-wave theory[6] uses a basic set of states differing from (15). The basic states are defined so as to be rigorously orthogonal to each other, so that the kinematical spin-wave interaction does not appear. The total interaction is then a dynamical interaction given by an equation similar to Eq. (26), but with coefficients Q_{ba} which do not tend to zero at long wavelengths. Although the absence of kinematical interaction makes the situation superficially simpler, the resulting strong dynamical interaction leads to an essentially nonlinear and mathematically intractable formalism.

In our treatment of the theory, a tremendous simplification appears when we consider ideal scattering processes. An ideal scattering process is defined as a process in which a finite number of spin waves interact in an infinite and otherwise empty lattice, the spin waves in initial and final states being spatially separated from each other. This is precisely the type of scattering process to which the S-matrix analysis of quantum electrodynamics[9] can be applied. In such a process, the initial and final states are automatically orthogonal and unaffected by the kinematical interaction. The development of the system at intermediate times is governed by the Hamiltonian (26), and the nonorthogonality of the states $|a\rangle$ and $|b\rangle$ does not change the formal solution of the Schrödinger equation with this Hamiltonian. Therefore we reach the conclusion that *the kinematical interaction produces no effect at all on the matrix elements for ideal scattering processes.* We may calculate such processes with the dynamical interaction alone.

In Sec. 6 we shall calculate in detail the matrix elements for scattering of one spin wave by another. We find, as expected from the smallness of Eq. (27), that the scattering becomes very small at long wavelengths. This shows apodictically that the large interaction of the Holstein-Primakoff treatment is illusory. If in fact the equations of Holstein-Primakoff could be solved exactly, they must lead to the same small scattering matrix elements that we calculate. The large effect of the diagonal matrix elements of the Holstein-Primakoff interaction must be almost exactly compensated by effects of the nondiagonal matrix elements which are customarily neglected.

We may state quite generally that the interaction between two isolated spin waves is *only* the dynamical interaction. The kinematical interaction is a purely statistical effect which reduces the statistical weight of states containing a high density of spin waves per unit volume. The kinematical interaction therefore appears in calculations of the statistical and thermodynamical properties of the spin-wave system, but not in the dynamics of individual spin waves.

There exists a naive picture of the kinematical interaction[10] in the case $S=\frac{1}{2}$, according to which one spin wave creates an "obstacle" to the passage of a second spin wave through the same region of space, because two reversed spins cannot occupy the same atom simultaneously. This picture would lead one to expect a scattering cross section like that of a "hard sphere" of atomic dimensions. In fact, the picture is totally incorrect. Two spin waves of long wavelength can "float over" each other almost without mutual interference, even in the case $S=\frac{1}{2}$.

4. PARTITION FUNCTION EXPRESSED IN TERMS OF NONORTHOGONAL STATES

The partition function of the model is

$$Z=\text{Spur}[\exp(-\beta\mathfrak{H})], \quad \beta=(kT)^{-1}. \quad (28)$$

As is well known, the spur may be evaluated by ex-

[10] This picture seems to be responsible for the large interaction effects calculated by Van Kranendonk in reference 7.

pressing it as a sum over any complete orthogonal set of states. Our problem is that we have a convenient and approximately diagonal representation of $\mathfrak{K}$ given by Eq. (26), only in terms of the nonorthogonal spin-wave states $|a\rangle$. We shall now develop a method for calculating Z by means of sums over nonorthogonal states.

The starting point is the observation that the states $|a\rangle$ are redundant, and certainly not deficient, for spanning the Hilbert space of the system. Also these states appear to be distributed in a rather symmetrical way over the Hilbert space. Hence we may expect that the density matrix

$$K=\sum_a |a\rangle\langle a| \tag{29}$$

is a fairly good approximation to the unit operator, and will differ from the unit operator only by giving different statistical weight to different parts of the Hilbert space. We shall confirm this guess by calculating K exactly.

A complete and orthogonal, but not normalized, set of states for the system is defined by

$$|u\rangle=\prod_j[(2S)^{-\frac{1}{2}u_j}(u_j!)^{-\frac{1}{2}}(S_j^+)^{u_j}]|0\rangle, \tag{30}$$

where (u) represents a set of integers u_j, one attached to each lattice point $\mathbf{j}$, each taking only the values 0, 1, $\cdots$, $2S$. These are the states in which each spin individually has a definite orientation. The normalization is easily found from Eq. (3):

$$\langle v|u\rangle=F_u\delta_{uv}, \quad F_u=\prod_j F(u_j), \tag{31}$$

$$F(u)=1\cdot\left(1-\frac{1}{2S}\right)\cdots\left(1-\frac{u-1}{2S}\right). \tag{32}$$

From Eqs. (10), (15), and (30) it follows that the states $|a\rangle$ and $|u\rangle$ are orthogonal unless

$$\sum_j u_j=\sum_\lambda a_\lambda=q. \tag{33}$$

When Eq. (33) is satisfied, we write $\mathbf{j}_1, \cdots, \mathbf{j}_q$ for the indices $\mathbf{j}$ appearing in (u), taken in any order and with the appropriate multiplicities u_j. Similarly we write $\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q$ for the indices $\boldsymbol{\lambda}$ appearing in (a). Then Eqs. (10), (15), and (30) give

$$\langle u|a\rangle=[\prod_j(u_j!)^{-\frac{1}{2}}][\prod_\lambda(a_\lambda!)^{-\frac{1}{2}}]N^{-\frac{1}{2}q} \times F_u\sum_P\exp[i\sum_r\boldsymbol{\lambda}_r\cdot P\mathbf{j}_r], \tag{34}$$

where $P\mathbf{j}_r$ denotes any permutation of the indices $\mathbf{j}_1, \cdots, \mathbf{j}_q$, and the sum is over all of the $(q!)$ such permutations.

Now consider the expression

$$\langle u|K|v\rangle=\sum_a\langle u|a\rangle\langle a|v\rangle. \tag{35}$$

The summation over (a) is equivalent to a summation over each index $\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q$ over the whole reciprocal lattice independently, with a factor $[\prod_\lambda(a_\lambda)!/q!]$ to correct for the fact that all permutations of $\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q$ give the same (a). When Eq. (34) is substituted into (35), the summations over $\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_q$ can be performed immediately, giving the result zero unless the indices $\mathbf{j}_1, \cdots, \mathbf{j}_q$ appearing in (u) and (v) are identical apart from a permutation. When the result is not zero, the number of permutations contributing to (35) is $[\prod_j(u_j)!q!]$. Hence (35) reduces to

$$\langle u|K|v\rangle=F_u^2\delta_{uv}. \tag{36}$$

Since the states $|u\rangle$ are a complete set, Eqs. (31) and (36) imply that

$$K=\sum_a|a\rangle\langle a|=\sum_u|u\rangle\langle u|. \tag{37}$$

This verifies the statement that K is not very different from the unit operator. According to Eq. (31) the unit operator may be written

$$I=\sum_u F_u^{-1}|u\rangle\langle u|, \tag{38}$$

so that I and K are identical when $S=\frac{1}{2}$. Also Eq. (37) implies that

$$K^{-1}=\sum_u F_u^{-2}|u\rangle\langle u|. \tag{39}$$

Using Eqs. (37) and (39), we have an exact formula for the partition function:

$$Z=\mathrm{Spur}[\exp(-\beta\mathfrak{K})KK^{-1}] =\sum_u F_u^{-2}\sum_a\langle a|u\rangle\langle u|\exp(-\beta\mathfrak{K})|a\rangle, \tag{40}$$

in which $\mathfrak{K}$ operates only on spin-wave states $|a\rangle$.

5. TRANSITION TO IDEAL SPIN WAVES

The partition function of our model will now be interpreted in terms of another model which we call the ideal spin-wave model. In the ideal model, to each lattice site $\mathbf{j}$ is attached a harmonic oscillator which has states labeled by an integer u_j taking values from 0 to ∞. The oscillators possess creation operators η_j^* and annihilation operators η_j satisfying the relations

$$[\eta_j,\eta_k]=[\eta_j^*,\eta_k^*]=0, \quad [\eta_j,\eta_k^*]=\delta_{jk}, \tag{41}$$

$$\eta_j^*\eta_j=u_j. \tag{42}$$

A complete set of states for the whole system is

$$|u\rangle=\prod_j[(u_j!)^{-\frac{1}{2}}(\eta_j^*)^{u_j}]|0\rangle. \tag{43}$$

These are not only orthogonal but correctly normalized.

In the ideal model, we can define another complete orthogonal set of states:

$$|a\rangle=\prod_\lambda[(a_\lambda!)^{-\frac{1}{2}}(\alpha_\lambda^*)^{a_\lambda}]|0\rangle, \tag{44}$$

where α_λ^* is defined by

$$\alpha_\lambda^*=N^{-\frac{1}{2}}\sum_j\exp(i\boldsymbol{\lambda}\cdot\mathbf{j})\eta_j^*. \tag{45}$$

The α_λ^* are creation operators for harmonic oscillators whose states are labeled by the integers a_λ, and they satisfy the relations

$$[\alpha_\lambda,\alpha_\mu^*]=\delta_{\lambda\mu}, \quad \alpha_\lambda^*\alpha_\lambda=a_\lambda. \tag{46}$$

These harmonic oscillators, one attached to each point of the reciprocal lattice, we call "ideal spin waves."

The physical spin-wave states $|a\rangle$ and the ideal

spin-wave states $|a)$ are in one-to-one correspondence, but they belong to totally different Hilbert spaces. The states $|a)$ are orthogonal and kinematically independent, while the states $|a\rangle$ are not. After we have expressed the partition function (40) in terms of the ideal states alone, we shall be able to carry through the whole subsequent calculation with the ideal model, treating the spin waves as rigorously independent oscillators.

We first construct an operator $\mathfrak{K}$ in the ideal Hilbert space which has the same effect on the ideal spin-wave states $|a)$ as the Hamiltonian $\mathfrak{K}$ in the physical Hilbert space has on the states $|a\rangle$. That is to say, we require

$$\mathfrak{K}|a)=[E_0+\textstyle\sum_\lambda a_\lambda(L+\epsilon_\lambda)]|a)+\sum_b Q_{ba}|b), \quad (47)$$

with the same numerical coefficients ϵ_λ and Q_{ba} which appear in Eq. (26). This condition is satisfied by choosing

$$\mathfrak{K}=E_0+\textstyle\sum_\lambda(L+\epsilon_\lambda)\alpha_\lambda^*\alpha_\lambda -\frac{1}{4}JN^{-1}\sum_{\lambda\rho\sigma}\alpha_{\sigma+\lambda}^*\alpha_{\rho-\lambda}^*\alpha_\rho\alpha_\sigma\Gamma_{\rho\sigma}{}^\lambda. \quad (48)$$

To see that Eq. (48) leads to Eq. (47), it is only necessary to compute the double commutator

$$[[\mathfrak{K},\alpha_\rho^*],\alpha_\sigma^*]=-\tfrac{1}{2}JN^{-1}\textstyle\sum_\lambda\Gamma_{\rho\sigma}{}^\lambda\alpha_{\sigma+\lambda}^*\alpha_{\rho-\lambda}^*; \quad (49)$$

the deduction of Eq. (47) from (49) is identical with the deduction of Eq. (26) from (22).

Suppose next that the exponentiation of Eq. (47) leads to a linear relation

$$\exp[-\beta(\mathfrak{K}-E_0)]|a)=\textstyle\sum_b U_{ba}|b), \quad (50)$$

the coefficients U_{ba} being matrix elements

$$U_{ba}=(b|\exp[-\beta(\mathfrak{K}-E_0)]|a). \quad (51)$$

Then Eq. (26) will imply a relation

$$\exp[-\beta(\mathfrak{K}-E_0)]|a\rangle=\textstyle\sum_b U_{ba}|b\rangle, \quad (52)$$

with the same coefficients U_{ba} given by Eq. (51). Note that the analog of Eq. (51) with $|a\rangle$ replacing $|a)$ is false. Using Eq. (52), the partition function (40) may be written

$$Z=e^{-\beta E_0}\textstyle\sum_u F_u^{-2}\sum_{ab}U_{ba}\langle a|u\rangle\langle u|b\rangle. \quad (53)$$

Comparing the expression (34) for the scalar product $\langle u|a\rangle$ with the analogous expression obtained from Eqs. (43), (44), and (45), we find

$$\langle u|a\rangle=F_u(u|a). \quad (54)$$

Hence Eq. (53) becomes

$$\begin{aligned}Z&=\textstyle\sum_{uab}E_u(u|b)(b|\exp(-\beta\mathfrak{K})|a)(a|u)\\&=\textstyle\sum_u E_u(u|\exp(-\beta\mathfrak{K})|u),\end{aligned} \quad (55)$$

where

$$\begin{aligned}&E_u=\textstyle\prod_j E(u_j),\\&E(u)=1 \quad \text{for} \quad u=0,\,1,\,\cdots,\,2S,\\&E(u)=0 \quad \text{for} \quad u>2S.\end{aligned} \quad (56)$$

The factor E_u is essential here in order to take account of the fact that the summation in Eq. (53) extends only over physical states $|u\rangle$, which means that each u_j must not exceed $2S$.

The formula (55) with $\mathfrak{K}$ given by Eq. (48) expresses the partition function entirely in terms of the ideal harmonic oscillator model. The subsequent calculations are all based on Eqs. (48) and (55), so that from this point onward the physical model need not be mentioned. In Eq. (55) the kinematical spin-wave interaction is fully described by the factor E_u, and it is clear that this interaction will become rapidly less important as the value of S increases. The dynamical interaction is contained in the last term of (48), which describes a mutual scattering of two Bose particles with conservation of momentum. It is convenient that the two interactions are clearly separated and can be handled by different methods.

The operator $\mathfrak{K}$ takes a particularly simple form when written in terms of the atomic oscillator coordinates (41), namely

$$\mathfrak{K}=E_0+L\textstyle\sum_j\eta_j^*\eta_j+\frac{1}{2}JS\sum_{j\delta}(\eta_j^*-\eta_{j+\delta}^*)(\eta_j-\eta_{j+\delta}) +\frac{1}{4}J\sum_{j\delta}\eta_j^*\eta_{j+\delta}^*(\eta_j-\eta_{j+\delta})^2. \quad (57)$$

This shows that the dynamical interaction in the ideal model still acts only between nearest neighbors. The last term in Eq. (57) is not Hermitian, and so $\mathfrak{K}$ cannot be directly interpreted as a Hamiltonian in the ideal model, as it could in the physical model. But this gives rise to no difficulties in practical calculations. In particular, the argument at the end of Sec. 3 proves that we may calculate all matrix elements of scattering processes in the usual way, treating $\mathfrak{K}$ as if it were an ordinary Hamiltonian, and ignoring the distinction between physical and ideal models.

6. EIGENSTATES OF THE DYNAMICAL INTERACTION

The Hamiltonian $\mathfrak{K}$ will be written

$$\mathfrak{K}=E_0+H_m+H_1+H_2, \quad (58)$$

where H_2 is the last term in Eq. (48) or (57) and represents the dynamical interaction, while

$$H_m|a)=L\textstyle\sum_\lambda a_\lambda|a), \quad (59)$$

$$H_1|a)=\epsilon_a|a), \quad \epsilon_a=\textstyle\sum_\lambda a_\lambda\epsilon_\lambda. \quad (60)$$

To understand completely the effects of the dynamical interaction, we shall need to evaluate the matrix elements U_{ba} defined by Eq. (51). The problem is greatly simplified by the fact that every term in $\mathfrak{K}$ conserves the total number of particles $\sum a_\lambda$, so that the only nonvanishing U_{ba} are those for which

$$\textstyle\sum_\lambda a_\lambda=\sum_\lambda b_\lambda=q. \quad (61)$$

When $q=0$ or $q=1$ the solution is trivial, because then $H_2|a)=0$ and Eqs. (59) and (60) give

$$U_{ba}=\exp[-\beta(Lq+\epsilon_a)]\delta_{ba}. \quad (62)$$

In later sections we shall attack with full generality the problem of calculating U_{ba} for any value of q. In this section we study in some detail the mutual scattering of two spin waves, or in other words the behavior of the eigenstates of $\mathcal{H}$ in the case $q=2$. We shall use simple arguments, in order to gain a physical understanding of the effects of the dynamical interaction, before embarking on the heavy mathematics of the later analysis. This section is to be compared with the treatment by Bethe[3] of the corresponding problem in one dimension.

A general 2-particle state may be written in the form

$$\psi=\sum_{jk}\psi(\mathbf{j},\mathbf{k})\eta_j^*\eta_k^*|0), \tag{63}$$

where $\psi(\mathbf{j},\mathbf{k})=\psi(\mathbf{k},\mathbf{j})$ is a conventional 2-particle wave function in the lattice space, with the normalization condition

$$|\psi|^2=2\sum_{jk}|\psi(\mathbf{j},\mathbf{k})|^2=1. \tag{64}$$

Operating on such a state, the Hamiltonian H_1 gives

$$H_1\psi=JS\sum_{jk}\eta_j^*\eta_k^*|0) \times[\sum_\delta(2\psi(\mathbf{j},\mathbf{k})-\psi(\mathbf{j}+\boldsymbol{\delta},\mathbf{k})-\psi(\mathbf{j},\mathbf{k}-\boldsymbol{\delta})], \tag{65}$$

while the dynamical interaction H_2 gives

$$H_2\psi=J\sum_{j\delta}\eta_j^*\eta_{j+\delta}^*|0)[\psi(\mathbf{j},\mathbf{j})-\psi(\mathbf{j},\mathbf{j}+\boldsymbol{\delta})]. \tag{66}$$

A state $|a)$, containing two noninteracting particles with momenta $(\boldsymbol{\lambda}+\boldsymbol{\mu})$ and $(\boldsymbol{\lambda}-\boldsymbol{\mu})$, has the wave function (not normalized)

$$\psi_a(\mathbf{j},\mathbf{k})=\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]\cos[\boldsymbol{\mu}\cdot(\mathbf{j}-\mathbf{k})]. \tag{67}$$

We shall now construct a wave function ψ satisfying the Schrödinger equation

$$(H_1+H_2)\psi=(\mathcal{H}-E_0-2L)\psi=\epsilon_a\psi, \tag{68}$$

and representing the same two particles with interaction. This is strictly possible only in an infinite lattice, therefore all formulas in the present section are to be understood as valid in the limit as $N\to\infty$.

We define a Green's function for the Hamiltonian H_1 by the equation

$$(H_1-\epsilon_a)\{\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]G(\mathbf{k}-\mathbf{j})\} =2SJ\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]\delta(\mathbf{k}-\mathbf{j}). \tag{69}$$

The function $\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]G(\mathbf{k}-\mathbf{j})$ then represents noninteracting pairs of particles spreading out from a point source at the position $\mathbf{k}=\mathbf{j}$, with a well-defined total momentum $2\boldsymbol{\lambda}$. By taking the Fourier transform of Eq. (69), we obtain an explicit formula for $G(\mathbf{j})$:

$$G(\mathbf{j})=2SJN^{-1}\sum_\rho[\epsilon_{\lambda+\rho}+\epsilon_{\lambda-\rho}-\epsilon_a]^{-1}\exp(i\boldsymbol{\varrho}\cdot\mathbf{j}), \tag{70}$$

with the integration over the singularity defined in the usual way so that only outgoing waves exist at large distances. We assume for the wave function ψ which satisfies Eq. (68) the ansatz

$$\psi=\psi_a+\sum_\delta A_\delta\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]G(\mathbf{k}-\mathbf{j}-\boldsymbol{\delta}), \tag{71}$$

with coefficients A_δ still to be determined. This represents an incident plus a scattered wave, as in the usual treatments of scattering problems in quantum mechanics. By virtue of Eqs. (60) and (69), the Schrödinger equation (68) reduces to

$$H_2\psi+2SJ\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]\sum_\delta A_\delta\delta(\mathbf{k}-\mathbf{j}-\boldsymbol{\delta})=0. \tag{72}$$

From Eq. (66) we see that $H_2\psi(\mathbf{j},\mathbf{k})$ vanishes except when $(\mathbf{j},\mathbf{k})$ are nearest neighbors with $\mathbf{k}=\mathbf{j}+\boldsymbol{\delta}$. Therefore Eq. (68) is automatically satisfied for all $\mathbf{k}\neq\mathbf{j}+\boldsymbol{\delta}$, by virtue of the form assumed for the scattered wave. It remains only to satisfy Eq. (72) for $\mathbf{k}=\mathbf{j}+\boldsymbol{\delta}$ by a suitable choice of the A_δ.

We write $\mathbf{k}=\mathbf{j}+\boldsymbol{\Delta}$ in Eq. (72), where $\boldsymbol{\Delta}$ is a nearest neighbor lattice vector. Then Eq. (72) with (66) gives

$$\begin{aligned}2S&\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]A_\Delta\\ &=-\tfrac{1}{2}[\psi(\mathbf{j},\mathbf{j})+\psi(\mathbf{k},\mathbf{k})-\psi(\mathbf{j},\mathbf{k})-\psi(\mathbf{k},\mathbf{j})]\\ &=\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})]\{\cos(\boldsymbol{\mu}\cdot\boldsymbol{\Delta})-\cos(\boldsymbol{\lambda}\cdot\boldsymbol{\Delta})\\ &\quad+\sum_\delta A_\delta[\tfrac{1}{2}G(\boldsymbol{\delta}-\boldsymbol{\Delta})+\tfrac{1}{2}G(\boldsymbol{\delta}+\boldsymbol{\Delta})\\ &\qquad-\cos(\boldsymbol{\lambda}\cdot\boldsymbol{\Delta})G(\boldsymbol{\delta})]\}.\end{aligned} \tag{73}$$

This is a finite set of linear equations from which the A_δ can be determined.

Now we make an approximation which becomes very good when the particles in the state $|a)$ have wavelengths long compared with $\boldsymbol{\delta}$, i.e., when $(\boldsymbol{\lambda}\cdot\boldsymbol{\delta})$ and $(\boldsymbol{\mu}\cdot\boldsymbol{\delta})$ are small. This will always be valid at low temperatures. The function $G(\mathbf{j})$ appears in Eq. (73) only with $\mathbf{j}$ equal to a nearest neighbor or next-nearest neighbor lattice vector. For such small values of $\mathbf{j}$, the main contribution to the sum (70) comes from large $\boldsymbol{\varrho}$, and it therefore makes little difference if we set $\boldsymbol{\lambda}=\boldsymbol{\mu}=0$ in Eq. (70). Of course, it is still essential to use the correct Eq. (70) when discussing the behavior of the wave functions (71) at distances $|\mathbf{k}-\mathbf{j}|$ large compared with δ. But in Eq. (73) we may replace $G(\mathbf{j})$ by the function

$$G_0(\mathbf{j})=N^{-1}\sum_\rho[\gamma_0-\gamma_\rho]^{-1}\exp(i\boldsymbol{\varrho}\cdot\mathbf{j}), \tag{74}$$

with a relative error which tends to zero at low temperatures.

The function $G_0(\mathbf{j})$ has a simple classical interpretation. It satisfies the difference equation

$$\sum_\delta[G_0(\mathbf{j})-G_0(\mathbf{j}+\boldsymbol{\delta})]=\delta(\mathbf{j}). \tag{75}$$

Imagine an electrical network with a junction at each lattice point and a wire of unit conductivity joining each pair of nearest neighbors. Suppose a unit current is fed into the network at the junction O. The lattice is considered infinite so that the current flows out to zero potential at infinity. The potential at the junction $\mathbf{j}$ is determined by Eq. (75), and is therefore equal to $G_0(\mathbf{j})$.

In addition to replacing G by G_0, we replace $\cos(\boldsymbol{\lambda}\cdot\boldsymbol{\Delta})$ by 1 in the last term of Eq. (73), and in the other term write

$$\cos(\boldsymbol{\lambda}\cdot\boldsymbol{\Delta})-\cos(\boldsymbol{\mu}\cdot\boldsymbol{\Delta})=-\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta})(\boldsymbol{\tau}\cdot\boldsymbol{\Delta}), \tag{76}$$

where $\boldsymbol{\sigma}=\boldsymbol{\lambda}+\boldsymbol{\mu}$, $\boldsymbol{\tau}=\boldsymbol{\lambda}-\boldsymbol{\mu}$ are the momenta of the incident particles in the state $|a)$. These approximations are all good at long wavelengths. Equation (73) then becomes

$$2SA_\Delta+\sum_\delta A_\delta[G_0(\boldsymbol{\delta})-\tfrac{1}{2}G_0(\boldsymbol{\delta}-\boldsymbol{\Delta})-\tfrac{1}{2}G_0(\boldsymbol{\delta}+\boldsymbol{\Delta})] =\tfrac{1}{2}(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta})(\boldsymbol{\tau}\cdot\boldsymbol{\Delta}). \quad (77)$$

The solution of Eq. (77) for a lattice of any symmetry type may be written in the form

$$A_\Delta=\sum_1^5 A_iY_i(\boldsymbol{\sigma},\boldsymbol{\tau})Y_i(\boldsymbol{\Delta})+A'(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})\Delta^2, \quad (78)$$

where the Y_i are 5 orthogonal second-order spherical harmonics, each belonging to some irreducible representation of the lattice group, and the A_i are equal for Y_i belonging to the same representation. For the cubic group there are two irreducible representations by second-order harmonics, with dimensions 3 and 2. Accordingly the general form of Eq. (78) for a cubic lattice is

$$A_\Delta=A_1Z_1(\boldsymbol{\Delta})+A_2Z_2(\boldsymbol{\Delta})+A'(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})\Delta^2, \quad (79)$$

$$Z_1(\boldsymbol{\Delta})=(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta})(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta})-\sum_{123}\sigma_i\tau_i\Delta_i^2, \quad (80)$$

$$Z_2(\boldsymbol{\Delta})=\sum_{123}\sigma_i\tau_i\Delta_i^2-\tfrac{1}{3}(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})\Delta^2, \quad (81)$$

with three independent coefficients A_1, A_2, A'. When Eq. (79) is substituted into (77), the result is

$$A'=(12S)^{-1},\quad A_i=\tfrac{1}{2}[2S-\Gamma_i]^{-1},\quad i=1,2, \quad (82)$$

where the Γ_i are constants, depending only on the geometry of the lattice and defined by the relations

$$\sum_\delta Z_i(\boldsymbol{\delta})G_0(\boldsymbol{\delta}+\boldsymbol{\Delta})=\Gamma_iZ_i(\boldsymbol{\Delta}),\quad i=1,2. \quad (83)$$

We shall now determine these constants explicitly for the three types of cubic lattice.

If the lattice is simple cubic Z_1 vanishes identically, and if it is body-centered cubic Z_2 vanishes. In either case, Eq. (79) reduces to the form

$$A_\Delta=\tfrac{1}{2}[2S-\Gamma]^{-1}[(\boldsymbol{\sigma}\cdot\boldsymbol{\Delta})(\boldsymbol{\tau}\cdot\boldsymbol{\Delta})-\tfrac{1}{3}(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})\Delta^2] +(12S)^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})\Delta^2. \quad (84)$$

Equation (83) gives, for the simple cubic lattice,

$$\Gamma^s=6[G_0(100)-G_0(110)] =\tfrac{1}{4}-\tfrac{3}{2}[G_0(100)-G_0(200)]. \quad (85)$$

and for the body-centered

$$\Gamma^b=4[2G_0(111)-G_0(200)-G_0(220)] =\tfrac{1}{6}-(4/3)[G_0(111)-G_0(222)]. \quad (86)$$

For the face-centered cubic lattice neither term in Eq. (79) vanishes. In this case the two constants Γ_i in Eq. (83) are given by

$$\Gamma_1{}^f=4[2G_0(110)-G_0(200)-G_0(211)] =\tfrac{1}{6}-4[G_0(110)-G_0(211)] -2[G_0(110)-G_0(220)]. \quad (87)$$

$$\Gamma_2{}^f=6[G_0(110)-G_0(211)] =\tfrac{1}{8}-3[G_0(110)-G_0(200)] -\tfrac{3}{2}[G_0(110)-G_0(220)]. \quad (88)$$

In the electrical analog interpretation of Eq. (75), every one of the quantities in square brackets in Eqs. (85)–(88) is a potential difference between a nearest neighbor of O and a junction more distant from O. Therefore all these quantities in square brackets are positive, and we have proved without any numerical work the inequalities

$$\begin{aligned}&0<\Gamma^s<\tfrac{1}{4},\quad 0<\Gamma^b<\tfrac{1}{6},\\ &0<\Gamma_1{}^f<\tfrac{1}{6},\quad 0<\Gamma_2{}^f<\tfrac{1}{8}.\end{aligned} \quad (89)$$

Precise values for the constants may be obtained from Eq. (74) after converting the sum into an integral. Equation (74) gives

$$G_0(\mathbf{j})=\int_{-\pi}^{\pi}\int\int Q^{-1}\cos(j_1x+j_2y+j_3z)dxdydz, \quad (90)$$

where for the simple cubic lattice

$$Q=16\pi^3(3-\cos x-\cos y-\cos z), \quad (91)$$

$$\Gamma^s=6\int_{-\pi}^{\pi}\int\int Q^{-1}\cos x(1-\cos y)dxdydz\approx\tfrac{1}{5}. \quad (92)$$

For the face-centered cubic,

$$Q=32\pi^3(3-\cos x\cos y-\cos y\cos z-\cos z\cos x), \quad (93)$$

$$\Gamma_1{}^f=4\int_{-\pi}^{\pi}\int\int Q^{-1}[\cos x\cos y(2-\cos 2z) -\cos 2z]dxdydz\approx\tfrac{1}{12}, \quad (94)$$

$$\Gamma_2{}^f=6\int_{-\pi}^{\pi}\int\int Q^{-1}\cos x\cos y(1-\cos 2z)dxdydz\approx\tfrac{1}{12}. \quad (95)$$

For the body-centered cubic,

$$Q=64\pi^3(1-\cos x\cos y\cos z), \quad (96)$$

$$\Gamma^b=4\int_{-\pi}^{\pi}\int\int Q^{-1}[2\cos x\cos y\cos z -\cos 2x(1+\cos 2y)]dxdydz\approx\tfrac{1}{8}. \quad (97)$$

To summarize the results of this section so far, the 2-particle eigenstates of $\mathfrak{H}$ are given by Eq. (71), with the Green's function $G(\mathbf{j})$ given by Eq. (70) and the constants A_δ by Eqs. (79), (82) and (92)–(97). These are ordinary scattering states. The question now arises whether there can exist states of another kind, in which two particles are permanently bound together by the

dynamical interaction. Bethe[3] found that such states do exist in the one-dimensional theory.

Such a bound state, if any exists, must have a wave function of the form (71), without any incoming wave ψ_a. The Green's function (70) will be exponentially decreasing at large distances, having an energy below the threshold $2\epsilon_\lambda$ at which dissociation into two free particles can occur. Supposing the wavelength of the center-of-mass motion to be long compared with δ, we may as before approximate $G(\mathbf{j})$ by the function

$$G_p(\mathbf{j})=N^{-1}\sum_\rho[\gamma_0-\gamma_\rho+p]^{-1}\exp(i\boldsymbol{\varrho}\cdot\mathbf{j}) \quad (98)$$

in Eq. (73), where p is either zero or positive. This function satisfies instead of Eq. (75)

$$\sum_\delta[G_p(\mathbf{j})-G_p(\mathbf{j}+\boldsymbol{\delta})]+pG_p(\mathbf{j})=\delta(\mathbf{j}). \quad (99)$$

The coefficients A_δ now satisfy Eq. (77) without any inhomogeneous term on the right-hand side and with G_p replacing G_0. The argument leading to Eq. (82) gives

$$2S-\Gamma_i=0, \quad (100)$$

either for $i=1$ or $i=2$, as a necessary condition for a nonzero set of A_δ to exist. Here Γ_i is defined by Eq. (83) with G_p for G_0. Formulas for Γ_i analogous to Eqs. (85)–(88) are easily derived. But now the electrical interpretation of Eq. (99) identifies $G_p(\mathbf{j})$ with the potential at vertex $\mathbf{j}$ in a "leaky" network where every junction is connected separately to earth by a leak of conductivity p. The current in each link of the leaky network is obviously less than that in the original network. Therefore the inequalities (89) still hold for the new Γ_i, and Eq. (100) is impossible.

We have proved that no bound states exist for two spin waves traveling together with a wavelength l long compared with δ. The approximations made were of relative order $(\delta/l)^2$ at most, whereas the condition (100) for a bound state to exist fails by a factor greater than 4. There are therefore certainly no bound states until l becomes quite small, comparable with δ. It is likely that no bound states exist at any wavelength, but we leave this question open. An exact discussion of Eq. (73) for general values of $\boldsymbol{\lambda}$ would be necessary to settle the question. In any case, it is only the bound states at long wavelengths which might, if they had existed, have had an observable effect on the thermodynamic behavior of the spin-wave system at low temperatures.

The fact that the spin-wave interaction leads to the formation of bound states in one dimension but not in three is not unexpected. It is well known in elementary quantum mechanics that any attractive potential well will bind a particle in one dimension, while in 3 dimensions the potential has to exceed a critical strength to produce binding. The condition (100) is just a statement of the critical interaction strength necessary for binding in the 3-dimensional spin-wave system. The quantity $(2S)^{-1}$ is the "effective coupling constant" of the dynamical spin-wave interaction, since it appears in the Hamiltonian (57) as the ratio of the interaction term H_2 to the "kinetic energy" term H_1. The critical coupling constant for the simple cubic lattice is about 5, and this can never be reached since $2S$ must be an integer.

Our treatment would break down in one dimension because the integrals (91)–(97) would not converge. For a simple square lattice in 2 dimensions, the method succeeds and gives an effective Γ lying between 0 and $\frac{1}{2}$. So for spin waves, just as for the weak attractive potential in elementary quantum mechanics, there are no bound states in 2 dimensions.

The method of this section could in principle be applied in order to obtain an exact enumeration of the eigenstates of $\mathcal{H}$ describing the interaction of 3 spin waves. But the details would become formidably complicated even with $q=3$, and the method is evidently unsuited for discussing larger values of q. We therefore postpone the discussion of the simultaneous interaction of 3 or more spin waves until Sec. 9, when we shall return to it with more powerful methods.

7. SCATTERING OF TWO SPIN WAVES

As a simple application of the results of the last section, we shall calculate the cross section for the scattering of two spin waves with wave vectors $\boldsymbol{\sigma}=\boldsymbol{\lambda}+\boldsymbol{\mu}$, $\boldsymbol{\tau}=\boldsymbol{\lambda}-\boldsymbol{\mu}$ into a state with wave vectors $\boldsymbol{\lambda}+\boldsymbol{\varrho}$, $\boldsymbol{\lambda}-\boldsymbol{\varrho}$. All wavelengths are assumed to be long compared with $\boldsymbol{\delta}$.

When the separation $r=|\mathbf{j}-\mathbf{k}|$ is large, the outgoing-wave Green's function (70) takes the asymptotic form

$$G(\mathbf{j}-\mathbf{k})=(3V/2\pi\gamma_0\delta^2)[e^{i\mu r}/r], \quad (101)$$

independent of λ. The wave function (71) becomes in the asymptotic region

$$\psi=\exp[i\boldsymbol{\lambda}\cdot(\mathbf{j}+\mathbf{k})][\cos(\boldsymbol{\mu}\cdot(\mathbf{j}-\mathbf{k}))+(V/4\pi)(e^{i\mu r}/r) \times\{(2S)^{-1}(\boldsymbol{\sigma}\cdot\boldsymbol{\tau})+X_1\delta^2A_1Z_1(\boldsymbol{\varrho})+X_2\delta^2A_2Z_2(\boldsymbol{\varrho})\}]. \quad (102)$$

Here $\boldsymbol{\varrho}$ is the vector with magnitude μ in the direction of $(\mathbf{j}-\mathbf{k})$, and A_1, A_2, $Z_1(\boldsymbol{\varrho})$, $Z_2(\boldsymbol{\varrho})$ are defined by Eqs. (80)–(82). The coefficients X_1, X_2 are given by

$$X_1=[0,\tfrac{1}{2},\tfrac{2}{3}], \quad X_2=[1,\tfrac{1}{4},0], \quad (103)$$

for the simple, face-centered, and body-centered lattices, respectively.

According to the discussion at the end of Sec. 3, the scattering cross section is given directly by the intensity of the scattered wave in Eq. (102). The three terms of the scattered wave constitute, respectively, S-waves, D-waves of the 3-dimensional representation of the cubic group, and D-waves of the 2-dimensional representation. Each term is given with a relative error tending to zero at long wavelengths. In the "near zone" of the wave function (71), all three terms are of the same order of magnitude. However, in the "wave zone" the D-waves are weaker by two powers of $(\mu\delta)$, which means that the D-wave terms in Eq. (102) are comparable with the error in the S-wave term.

Therefore to the leading order in $(\mu\delta)$ the scattering is pure S-wave and is isotropic in the "center-of-mass system." The total cross section is given by the S-wave component of Eq. (102), and is

$$\Sigma=[V^2/8\pi S^2]\sigma^2\tau^2\cos^2\phi, \tag{104}$$

where ϕ is the angle between the initial wave vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$. If $\boldsymbol{\sigma}'=\boldsymbol{\lambda}+\boldsymbol{\varrho}$ and $\boldsymbol{\tau}'=\boldsymbol{\lambda}-\boldsymbol{\varrho}$ are the two wave vectors of the final state and ϕ' the angle between them, then

$$\sigma\tau\cos\phi=\sigma'\tau'\cos\phi', \tag{105}$$

and so the cross section (104) is consistent with the principle of detailed balance. Since we have been using a non-Hermitian Hamiltonian, the achievement of detailed balance is not automatic and provides a check on the physical correctness of our method.

At long wavelengths, the energy ϵ_σ of a spin wave with wave vector $\boldsymbol{\sigma}$ becomes

$$\epsilon_\sigma=\tfrac{1}{6}JS\gamma_0\sigma^2\delta^2, \tag{106}$$

according to Eqs. (17) and (24). It is convenient to introduce the geometrical factor

$$\nu=\delta^2V^{-\frac{1}{3}}=[1,2^{\frac{1}{3}},3\times2^{-4/3}] \tag{107}$$

for the simple, face-centered, and body-centered lattices, respectively, and to define temperature by the dimensionless ratio

$$\theta=3kT/(2\pi JS\gamma_0\nu). \tag{108}$$

Then

$$(\epsilon_\sigma/kT)=\sigma^2V^{\frac{1}{3}}/(4\pi\theta), \tag{109}$$

and the cross section (104) becomes

$$\Sigma=2\pi V^{\frac{1}{3}}\theta^2S^{-2}(\epsilon_\sigma\epsilon_\tau/k^2T^2)\cos^2\phi. \tag{110}$$

Now the Bloch theory[1] gives, for the mean density of spin-wave energy at temperature T,

$$\bar{\epsilon}=\tfrac{3}{2}\zeta(\tfrac{3}{2})\theta^{\frac{3}{2}}(kT/V). \tag{111}$$

The reciprocal of the mean free path between collisions, for a spin wave with energy ϵ_σ traveling through a lattice in thermal equilibrium at temperature T, is given by substituting $\bar{\epsilon}$ for ϵ_τ in Σ and averaging over ϕ. Thus the mean free path is

$$\lambda_c=[\pi\zeta(\tfrac{3}{2})(\epsilon_\sigma/kT)]^{-1}S^2V^{\frac{1}{3}}\theta^{-7/2}. \tag{112}$$

For a given spin wave with $\boldsymbol{\sigma}$ independent of T, this varies with temperature like $T^{-5/2}$. For an average spin wave in thermal equilibrium, the mean free path is proportional to $T^{-7/2}$.

8. CONVERGENCE OF THE BORN APPROXIMATION SERIES

Instead of calculating the wave function (71) in closed form, we could have carried through a Born approximation treatment of the scattering of two spin waves, taking H_1 as the unperturbed Hamiltonian and H_2 as the perturbation. Because the quantity $(2S)^{-1}$ appears only in the ratio of H_2 to H_1, the contribution of the mth Born approximation to the scattering would be just the mth term of an expansion of the exact solution in powers of $(2S)^{-1}$. In the exact solution, the only place where $(2S)^{-1}$ appears is in the coefficients A_i, and according to Eq. (82),

$$A_i=\tfrac{1}{2}\sum_{m=1}^{\infty}(\Gamma_i)^{m-1}(2S)^{-m}. \tag{113}$$

The Born approximation series is comfortably convergent because of the smallness of the Γ_i. This is, of course, connected with the nonexistence of bound states of two spin waves.

The dynamical interaction between two spin waves is according to Eq. (57) a nearest neighbor interaction, taking effect when the two reversed spins are at positions $\mathbf{j}$, $\mathbf{k}$, with $\mathbf{k}=\mathbf{j}+\boldsymbol{\delta}$. When two spins come together, they may interact first at $\mathbf{k}=\mathbf{j}+\boldsymbol{\delta}_1$, then propagate and interact again at another nearest-neighbor position $\mathbf{k}=\mathbf{j}+\boldsymbol{\delta}_2$, and so on. The mth Born approximation describes the process in which the interaction occurs just m times before the two spins finally separate. The exact solution describes these multiple interactions completely. The propagation distance between multiple interactions is of the order of δ and so the effect of retardation between interactions is negligible for wavelengths long compared with δ. The neglect of this retardation produces the approximation (77) and leads to the simplicity of the solution (82).

With neglect of retardation, the effect of multiple interactions is merely to multiply the first Born approximation scattering by a constant "enhancement factor" given by

$$[1-(\Gamma_i/2S)]^{-1}. \tag{114}$$

Because of the structure of the interaction, the S-wave is not enhanced at all, and the two kinds of D-waves in the face-centered lattice receive different enhancement factors. The enhancement is in all cases numerically small because the probability of escape after each interaction is greater than the probability of another interaction.

A more general proof of the convergence of the Born approximation series is the following. Any two-particle state may be written in the form

$$\psi=\textstyle\sum_{\lambda\mu}\psi(\boldsymbol{\lambda},\boldsymbol{\mu})\alpha_\lambda^*\alpha_\mu^*|0), \tag{115}$$

$$|\psi|^2=2\textstyle\sum_{\lambda\mu}|\psi(\boldsymbol{\lambda},\boldsymbol{\mu})|^2=1. \tag{116}$$

Equation (48) gives

$$H_1\psi=\textstyle\sum_{\lambda\mu}(\epsilon_\lambda+\epsilon_\mu)\psi(\boldsymbol{\lambda},\boldsymbol{\mu})\alpha_\lambda^*\alpha_\mu^*|0), \tag{117}$$

$$H_2\psi=-\tfrac{1}{2}JN^{-1}\textstyle\sum_{\lambda\mu\rho}\Gamma_{\lambda+\rho,\,\mu-\rho}{}^{\rho}\times\psi(\boldsymbol{\lambda}+\boldsymbol{\varrho},\boldsymbol{\mu}-\boldsymbol{\varrho})\alpha_\lambda^*\alpha_\mu^*|0). \tag{118}$$

Applying Cauchy's inequality to Eq. (118), we find

$$|H_2\psi|^2=\tfrac{1}{2}J^2N^{-2}\textstyle\sum_{\lambda\mu}|\sum_\rho\Gamma_{\lambda+\rho,\,\mu-\rho}{}^{\rho}\psi(\boldsymbol{\lambda}+\boldsymbol{\varrho},\boldsymbol{\mu}-\boldsymbol{\varrho})|^2 \le\tfrac{1}{2}J^2N^{-1}\textstyle\sum_{\lambda\mu}|\psi(\boldsymbol{\lambda},\boldsymbol{\mu})|^2\sum_\rho|\Gamma_{\lambda\mu}{}^{\rho}|^2. \tag{119}$$

By Eqs. (27) and (24),

$$\sum_\rho |\Gamma_{\lambda\mu}{}^\rho|^2 = 4N\sum_\delta [1-\cos(\boldsymbol{\delta}\cdot\boldsymbol{\lambda})][1-\cos(\boldsymbol{\delta}\cdot\boldsymbol{\mu})] \leq 2N(JS)^{-2}\epsilon_\lambda\epsilon_\mu \leq \tfrac{1}{2}N(JS)^{-2}(\epsilon_\lambda+\epsilon_\mu)^2; \quad (120)$$

hence Eq. (119) gives

$$|H_2\psi| \leq \tfrac{1}{2}S^{-1}[\sum_{\lambda\mu}(\epsilon_\lambda+\epsilon_\mu)^2|\psi(\boldsymbol{\lambda},\boldsymbol{\mu})|^2]^{\frac{1}{2}} = (8S^2)^{-\frac{1}{2}}|H_1\psi|. \quad (121)$$

The numerical factor $(8S^2)^{-\frac{1}{2}}$ does not have any special significance, and it could be improved by a more careful handling of the inequalities. The important point is that the factor is definitely less than unity even for $S=\frac{1}{2}$. Therefore the series expansion

$$(H_1+H_2)^{-1}\psi = \sum_{m=0}^{\infty}(-1)^m H_1^{-1}(H_2H_1^{-1})^m\psi \quad (122)$$

converges with uniform rapidity for every 2-particle state ψ. The convergence of Eq. (122) is precisely the meaning of the statement that the Born approximation series converges for all 2-particle processes.

It is not clear whether the convergence will extend to interactions of more than 2 spin waves. Since the interaction energy of q particles is at most proportional to $\frac{1}{2}q(q-1)$, while the kinetic energy is proportional to q, the ratio between them will increase with $(q-1)$. For $q=2$, the ratio between successive Born approximations is at most $[\Gamma/2S]$, or approximately $[10\,S]^{-1}$ for the simple cubic lattice. Therefore we may expect the Born approximation series to converge for any number of particles up to $10\,S$. It seems physically plausible that when a substantial number (say $10\,S$ or more) of reversed spins are close together, they may form a bound state with lower energy than the same number of free spin waves, and the Born approximation will then certainly fail.

In the following sections we shall use the Born approximation series freely. The justification for this is that the series can always in principle be replaced by the exact solution of an interaction problem involving a finite number of particles. And in practice we shall only be concerned with the case $q=2$ where the series is known to converge.

9. GREEN'S FUNCTIONS OF THE DYNAMICAL INTERACTION

We now return to the general analysis of the matrix elements U_{ba} defined by Eq. (51). It is convenient to study the quantity

$$U_{hf} = (0|\exp(\sum_\lambda h_\lambda\alpha_\lambda)\exp[-\beta(\mathcal{H}-E_0)] \times \exp(\sum_\lambda f_\lambda\alpha_\lambda{}^*)|0), \quad (123)$$

where f_λ and h_λ are undetermined parameters. By Eq. (44), U_{ba} is the coefficient of

$$\prod_\lambda[(b_\lambda!a_\lambda!)^{-\frac{1}{2}}h_\lambda{}^{b_\lambda}f_\lambda{}^{a_\lambda}] \quad (124)$$

in U_{hf}. Equation (123) is expanded in a Born approximation series by using the identity

$$\exp[-\beta(\mathcal{H}-E_0)] = \sum_{m=0}^{\infty}(-1)^m\int_0^\beta d\beta_1\cdots\int_0^{\beta_{m-1}}d\beta_m \times\{\exp[(\beta_1-\beta)\mathcal{H}_1]H_2\exp[(\beta_2-\beta_1)\mathcal{H}_1]H_2\cdots H_2 \times\exp(-\beta_m\mathcal{H}_1)\}, \quad (125)$$

with

$$\mathcal{H}_1 = \mathcal{H}-E_0-H_2 = H_m+H_1. \quad (126)$$

Each factor H_2 is a sum of products of particle creation and annihilation operators according to Eq. (48). After expanding H_2 in this way, each factor containing $\mathcal{H}_1$ becomes a c-number, because $\mathcal{H}_1$ is diagonal in the free-particle states.

Following the method of Wick,[11] the individual terms in Eq. (125) can be broken up into "normal products" in which creation operators lie to the left of annihilation operators. As a consequence we find

$$U_{hf} = \sum_G U_{hf}(G), \quad (127)$$

where $U_{hf}(G)$ is a contribution from a particular "Feynman graph" G. The graphs have a simpler structure than in quantum electrodynamics.[9] Each graph contains a certain number $(2q+m)$ of vertices, of which q are called initial, q are final, and m are intermediate. It contains $(q+2m)$ lines, each joining two vertices and each having a direction marked in it. An initial vertex has one outgoing line attached to it and none ingoing. A final vertex has only one ingoing line. An intermediate vertex has two ingoing and two outgoing lines.

The intermediate vertices are labeled from 1 to m, and to the vertex j is attached an integration variable β_j. The initial and final vertices are labeled from 1 to q. To each line is attached a reciprocal lattice vector $\boldsymbol{\lambda}$.

Given G, the contribution $U_{hf}(G)$ is built up from the following factors:

(a) an over-all factor

$$(q!)^{-2}(m!)^{-1}\exp(-\beta qL); \quad (128)$$

(b) corresponding to each intermediate vertex at which lines $\boldsymbol{\varrho}$, $\boldsymbol{\sigma}$ are ingoing and lines $\boldsymbol{\mu}$, $\boldsymbol{\tau}$ are outgoing, a factor

$$\tfrac{1}{2}JN^{-1}\delta(\boldsymbol{\mu}+\boldsymbol{\tau}-\boldsymbol{\varrho}-\boldsymbol{\sigma})[\Gamma_{\rho\sigma}{}^{\mu-\sigma}+\Gamma_{\rho\sigma}{}^{\tau-\sigma}]; \quad (129)$$

(c) corresponding to each initial or final vertex at which the line $\boldsymbol{\lambda}$ is incident, a factor f_λ or h_λ;

(d) corresponding to each line $\boldsymbol{\lambda}$ joining vertex r to vertex s, a factor

$$\theta(\beta_s-\beta_r)\exp[-\epsilon_\lambda(\beta_s-\beta_r)], \quad (130)$$

where $\theta(x)=1$ for $x>0$, $\theta(x)=0$ for $x<0$ [if the vertex r is initial, we write 0 for β_r in Eq. (130), and if vertex s is final we write β for β_s];

(e) an integration from 0 to β with respect to the

[11] G. C. Wick, Phys. Rev. **80**, 268 (1950).

variable β_r for each intermediate vertex, and a summation over the variable λ for each line;

(f) a factor $\frac{1}{2}$ for any pair of lines which have a common vertex at both ends.

Every graph G is composed of one or more connected parts, say n_1 identical parts G_1, n_2 identical parts G_2, and so on. The relation between $U_{hf}(G)$ and the $U_{hf}(G_r)$ is then

$$U_{hf}(G)=\prod_r\{(n_r!)^{-1}[U_{hf}(G_r)]^{n_r}\}, \tag{131}$$

the factor $(n_r!)^{-1}$ arising because a permutation of the labels of the vertices between the identical parts G_r does not give rise to distinct labelings of G. Summing Eq. (131) over all G gives by Eq. (127)

$$U_{hf}=\exp[\textstyle\sum_G U_{hf}(G)], \tag{132}$$

where the summation in the exponent is taken only over connected G. The effects of disconnected graphs are exactly taken care of by the exponentiation.

The exponent in Eq. (132) may be written

$$\sum_G U_{hf}(G)=\sum_{\lambda\mu}\Gamma_1(\lambda,\mathfrak{y})h_\lambda f_\mu + \sum_{\lambda_1\lambda_2\mu_1\mu_2}\Gamma_2(\lambda_1\lambda_2,\mathfrak{y}_1\mathfrak{y}_2)h_{\lambda_1}h_{\lambda_2}f_{\mu_1}f_{\mu_2}+\cdots, \tag{133}$$

where $\Gamma_q(\lambda_1\cdots\lambda_q,\ \mathfrak{y}_1\cdots\mathfrak{y}_q)$ is a sum of contributions from all connected graphs G with a given number q of initial and final vertices. The function Γ_q is called the q-particle Green's function of the dynamical interaction. It describes completely the behavior of a system of q spin-waves interacting with each other.

Each Green's function Γ_q can in principle be calculated as the solution of a q-particle Schrödinger equation which can be written down in closed form. For example, Γ_2 satisfies the equation

$$\begin{aligned}-(\partial/\partial\beta)\Gamma_2(\lambda_1\lambda_2,\mathfrak{y}_1\mathfrak{y}_2)&=(2L+\epsilon_{\lambda_1}+\epsilon_{\lambda_2})\Gamma_2(\lambda_1\lambda_2,\mathfrak{y}_1\mathfrak{y}_2)\\&-(\tfrac{1}{4}JN^{-1})\sum_{\nu_1\nu_2}\delta(\lambda_1+\lambda_2-\nu_1-\nu_2)\\&\times[\Gamma_{\nu_1\nu_2}{}^{\lambda_1-\nu_2}+\Gamma_{\nu_1\nu_2}{}^{\lambda_2-\nu_2}]\{\tfrac{1}{4}[\Gamma_1(\nu_1,\mathfrak{y}_1)\Gamma_1(\nu_2,\mathfrak{y}_2)\\&+\Gamma_1(\nu_1,\mathfrak{y}_2)\Gamma_1(\nu_2,\mathfrak{y}_1)]+\Gamma_2(\nu_1\nu_2,\mathfrak{y}_1\mathfrak{y}_2)\}.\end{aligned} \tag{134}$$

The equation for Γ_1 is trivial and has the solution

$$\Gamma_1(\lambda,\mathfrak{y})=\delta_{\lambda\mu}\exp[-\beta(L+\epsilon_\lambda)]. \tag{135}$$

The equation for Γ_q is always linear, with an inhomogeneous term which contains the Γ_p with $p<q$.

In practice the Γ_q are more conveniently computed by using the Born approximation, that is to say by calculating the contributions from connected graphs with q initial and final vertices. This method at least leads to simple and reliable estimates of the order of magnitude of the various terms. However, since for each $q\geqslant 2$ there is an infinite number of connected graphs, the practicability of the method depends on the convergence of the Born approximation series which was discussed in Sec. 8.

10. KINEMATICAL INTERACTION

According to Eq. (55), the complete partition function is

$$Z=e^{-\beta E_0}\textstyle\sum_{ab}U_{ba}V_{ab}, \tag{136}$$

$$V_{ab}=\textstyle\sum_u(a|u)E_u(u|b). \tag{137}$$

We shall now study the V_{ab}, which embody the effects of the kinematical interaction. The idea is to bring V_{ab} into a form similar to that obtained for U_{ba} in Sec. 9.

V_{ab} is the coefficient of Eq. (124) in the expression

$$V_{fh}=\textstyle\sum_u(0|\exp(\sum_\lambda f_\lambda\alpha_\lambda)|u)\times E_u(u|\exp(\sum_\lambda h_\lambda\alpha_\lambda{}^*)|0). \tag{138}$$

Now write

$$\begin{aligned}\phi_j&=N^{-\frac{1}{2}}\textstyle\sum_\lambda\exp(-i\lambda\cdot\mathbf{j})f_\lambda,\\ \chi_j&=N^{-\frac{1}{2}}\textstyle\sum_\lambda\exp(i\lambda\cdot\mathbf{j})h_\lambda,\end{aligned} \tag{139}$$

so that by Eq. (45)

$$\textstyle\sum_\lambda f_\lambda\alpha_\lambda=\sum_j\phi_j\eta_j,\quad \sum_\lambda h_\lambda\alpha_\lambda{}^*=\sum_j\chi_j\eta_j{}^*. \tag{140}$$

Then Eq. (138) becomes

$$V_{fh}=\textstyle\prod_j[\sum_u(0|\exp(\phi_j\eta_j)|u)\times E(u)(u|\exp(\chi_j\eta_j{}^*)|0)], \tag{141}$$

where the matrix elements refer to states of the single harmonic oscillator at the lattice site $\mathbf{j}$, and $E(u)$ is defined by Eq. (56). Now

$$(0|\exp(\phi_j\eta_j)|u)=(u!)^{-\frac{1}{2}}\phi_j{}^u, \tag{142}$$

and therefore Eq. (141) gives

$$\begin{aligned}V_{fh}&=\textstyle\prod_j[\sum_u E(u)(u!)^{-1}(\phi_j\chi_j)^u]\\&=\exp\left[\sum_j\sum_{n=1}^{\infty}e(n)(\phi_j\chi_j)^n\right].\end{aligned} \tag{143}$$

The coefficients $e(n)$ are here defined by

$$\sum_{n=1}^{\infty}e(n)y^n=\log\left[\sum_{n=0}^{2S}(n!)^{-1}y^n\right]. \tag{144}$$

The $e(n)$ begin with the values

$$\begin{aligned}e(1)=0,\quad e(2)=e(3)=\cdots=e(2S)=0,\\ e(2S+1)=-[(2S+1)!]^{-1},\end{aligned} \tag{145}$$

and they satisfy the recurrence relation

$$\sum_{r=n-2S}^{n}\frac{re(r)}{(n-r)!}=0,\quad n>2S. \tag{146}$$

In particular, we have

$$e(n)=[(-1)^{n-1}/n],\quad S=\tfrac{1}{2}, \tag{147}$$

$$e(n)=[(-1)^{n-1}/n]2^{1-\frac{1}{2}n}\cos(\tfrac{1}{4}\pi n),\quad S=1. \tag{148}$$

The exponent in Eq. (143) will be expressed, by analogy with Eq. (133), in the form

$$\sum_j \sum_{n=1}^{\infty} e(n)(\phi_j\chi_j)^n = \sum_{\lambda\mu} \Delta_1(\mathfrak{y},\lambda) f_\mu h_\lambda + \sum_{\lambda_1\lambda_2\mu_1\mu_2} \Delta_2(\lambda_1\lambda_2,\mathfrak{y}_1\mathfrak{y}_2) f_{\mu_1} f_{\mu_2} h_{\lambda_1} h_{\lambda_2} + \cdots. \quad (149)$$

The function $\Delta_q(\mathfrak{y}_1\cdots\mathfrak{y}_q, \lambda_1\cdots\lambda_q)$ is called the q-particle Green's function of the kinematical interaction. Using Eq. (139), we find the explicit form of Δ_q,

$$\Delta_q(\mathfrak{y}_1\cdots\mathfrak{y}_q,\lambda_1\cdots\lambda_q) = e(q) N^{1-q} \delta\left[\sum_{r=1}^{q} (\mathfrak{y}_r - \lambda_r)\right]. \quad (150)$$

In particular, by Eq. (145),

$$\Delta_1(\mathfrak{y},\lambda) = \delta_{\mu\lambda}, \quad \Delta_2 = \Delta_3 = \cdots = \Delta_{2S} = 0. \quad (151)$$

11. EXACT FORMULA FOR THE FREE ENERGY

We now put the results of Secs. 9 and 10 together into the partition function (136). The summation over states a, b may be written

$$\sum_{ab} \left[\prod_\lambda (b_\lambda! a_\lambda!)^{-1}\right] = \sum_q (q!)^{-2} \sum_{\lambda_1\cdots\lambda_q\mu_1\cdots\mu_q}, \quad (152)$$

where $\lambda_1, \cdots, \lambda_q$ are the wave vectors of the q particles in state a, $\mathfrak{y}_1, \cdots, \mathfrak{y}_q$ are the wave vectors in state b, and each λ_r and $\mathfrak{y}_r$ is summed independently over the reciprocal lattice. Next, by Eqs. (124), (132), and (133), U_{ba} may be written

$$U_{ba} = \left[\prod_\lambda (b_\lambda! a_\lambda!)^{-\frac{1}{2}}\right] \sum_{PQ} \sum_{n_1 n_2 \cdots} \times \left[\prod_i (n_i!)^{-1} \prod_1^{n_i} \Gamma_i(P\lambda, Q\mathfrak{y})\right]. \quad (153)$$

Here P is summed over the $(q!)$ permutations of $(\lambda_1, \cdots, \lambda_q)$, and Q over the permutations of $(\mathfrak{y}_1, \cdots, \mathfrak{y}_q)$. The n_i are summed over all integer values consistent with

$$\sum i n_i = q. \quad (154)$$

The meaning of $\Gamma_i(P\lambda, Q\mathfrak{y})$ is the following. The final product in Eq. (153) is a product of n_1 functions Γ_1, n_2 functions Γ_2, and so on. The arguments $\lambda_1, \cdots, \lambda_q$, $\mathfrak{y}_1, \cdots, \mathfrak{y}_q$ are to be distributed in some fixed way among these functions, and then permuted by means of the permutations P and Q. In the same way, Eqs. (143) and (149) lead to the result

$$V_{ab} = \left[\prod_\lambda (b_\lambda! a_\lambda!)^{-\frac{1}{2}}\right] \sum_{P'Q'} \sum_{m_1 m_2 \cdots} \times \left[\prod_i (m_i!)^{-1} \prod_1^{m_i} \Delta_i(Q'\mathfrak{y}, P'\lambda)\right]. \quad (155)$$

Inserting Eqs. (152), (154), and (155) into (136) gives

$$Z = e^{-\beta E_0} \sum_q \sum_{\lambda_1\cdots\lambda_q\mu_1\cdots\mu_q} \sum_{PQ} \times \sum_{n_1 n_2 \cdots} \sum_{m_1 m_2 \cdots} \left[\prod_i (n_i!)^{-1} \prod_1^{n_i} \Gamma_i(P\lambda, Q\mathfrak{y})\right] \times \left[\prod_i (m_i!)^{-1} \prod_1^{m_i} \Delta_i(\mathfrak{y},\lambda)\right]. \quad (156)$$

We have here replaced the summation variables λ, $\mathfrak{y}$ by $(P')^{-1}\lambda$, $(Q')^{-1}\mathfrak{y}$, so that the summations over P', Q' become trivial and merely cancelled the $(q!)^{-2}$ in Eq. (152).

The multiple sum in Eq. (156) will now be partially disentangled. We say that two of the factors Γ_i, Δ_j are *connected* if either (a) they possess an argument λ_r or $\mathfrak{y}_r$ in common, or (b) they belong to a chain such as Γ_i, Δ_k, Γ_l, Δ_j, in which each consecutive pair has an argument in common. The factors Γ_i and Δ_j can then be grouped together into blocks, so that factors within the same block are connected, while factors in different blocks are not. The summations over λ and $\mathfrak{y}$ in different blocks are independent, and Eq. (156) breaks up into an exponential series, the exponent of which is a sum over single blocks. Hence we find an explicit formula for the free energy per atom at temperature T,

$$A = -(kT/N)\log Z = (E_0/N) - (\beta N)^{-1} \sum_q \sum_{\lambda_1\cdots\lambda_q\mu_1\cdots\mu_q} \sum_{PQ} \times \sum_{n_1 n_2 \cdots} \sum_{m_1 m_2 \cdots} \left[\prod_i (n_i!)^{-1} \prod_1^{n_i} \Gamma_i(P\lambda, Q\mathfrak{y})\right] \times \left[\prod_i (m_i!)^{-1} \prod_1^{m_i} \Delta_i(\mathfrak{y},\lambda)\right]. \quad (157)$$

This looks almost identical with Eq. (156). The difference is that the sum over P and Q now extends only over permutations which connect all the factors Γ_i and Δ_i into one block. This makes every term in Eq. (157) tend to a finite limit as $N\to\infty$, while in Eq. (156) there was a mixture of terms proportional to various powers of N.

Consider for example the terms with $m_1 = n_1 = q$, $m_i = n_i = 0$ for $i > 1$. Because of Eqs. (135) and (151), these may be written

$$A_B = -(\beta N)^{-1} \sum_q q^{-1} \sum_\lambda \exp[-q\beta(L+\epsilon_\lambda)] = (\beta N)^{-1} \sum_\lambda \log\{1 - \exp[-\beta(L+\epsilon_\lambda)]\}. \quad (158)$$

This is the free energy of a perfect gas of noninteracting spin waves, each carrying the energy $(L+\epsilon_\lambda)$. It is the total free energy in the linear approximation of the Bloch theory.[1] The corrections arising from spin-wave interactions will be given directly by the various terms in Eq. (157) involving at least one Γ_i or Δ_i with $i > 1$.

With the derivation of Eq. (157) the purpose of the

present paper is achieved. We have a starting point for detailed calculations of the thermodynamic effects of spin-wave interactions. Such calculations will be carried through, and quantitative results obtained, in a following paper.

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Thermodynamic Behavior of an Ideal Ferromagnet*

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The free energy of an ideal Heisenberg-model ferromagnet is calculated as a power series in the temperature T, using the mathematical machinery developed in an earlier paper. The spontaneous magnetization in zero external field is given by

$$[M(T)/M(0)]=S-a_0\theta^{3/2}-a_1\theta^{5/2}-a_2\theta^{7/2}-a_3S^{-1}\theta^4+O(\theta^{9/2}).$$

Here θ is the temperature in dimensionless units, and a_0, a_1, a_2, a_3 are positive numerical coefficients which are computed for the three types of cubic crystal lattice. The first two terms are the result of the simple Bloch theory in which spin waves are treated as noninteracting Bose particles with constant effective mass. The a_1 and a_2 corrections come from the variation of effective mass with velocity. The a_3 term is the lowest order correction arising from interaction between spin waves. This result is in violent contradiction to earlier published calculations which gave interaction effects proportional to $T^{7/4}$ and T^2.

The smallness of the thermodynamic effects of spin-wave interactions is discussed in physical terms, and partially explained, in the introduction of this paper. A general proof is given that the thermodynamic effects of the "exclusion principle," which forbids more than $(2S)$ spin deviations to occupy the same atom, are of order $\exp(-a\theta^{-1})$ and give zero contribution to any finite power of θ. The residual dynamical interaction between 2 spin waves gives rise to a second virial coefficient b_2' which is calculated and shown to be of order $T^{5/2}$. The a_3 term in the magnetization is proportional to b_2'. Effects of interaction of 3 or more spin waves are estimated and found to be of order θ^5 or higher.

1. INTRODUCTION

IN the preceding paper,[1] a mathematical formalism was constructed to describe the motions of spin waves in an ideal ferromagnetic lattice. In particular, an exact formula (I, 157) was derived for the free energy of such a lattice. In this paper the free energy will be evaluated as a series expansion in powers of the temperature T. The results may be expected to provide an accurate description of the thermodynamics of the model in the range of low temperatures, say below one quarter of the Curie temperature. The notations and definitions of the earlier paper will be used without further explanation.

The quantity which is of the greatest practical interest is the spontaneous magnetization $M(T)$ of the lattice in zero external field. In the linear approximation of the Bloch spin-wave theory,[2] this magnetization is given by the formula

$$[M(T)/M(0)]=S-\zeta(\tfrac{3}{2})\theta^{3/2}. \quad (1)$$

Here $\zeta(a)=\sum_1^\infty n^{-a}$ is the Riemann zeta function, S is the spin of each atom, and θ is the dimensionless temperature defined by Eq. (I, 108). Considerable uncertainty has existed concerning the accuracy of this formula. Kramers and Opechowski[3] have calculated additional terms in an expansion in ascending powers of θ, and find the next term to be of order θ^2. However, Schafroth,[4] using the spin-wave formalism of Holstein and Primakoff,[4] finds a term in $\theta^{7/4}$ with a positive coefficient, which would interfere seriously with the $\theta^{3/2}$ term in the temperature range of current experiments.[5] Van Kranendonk[6] by another method arrives at a term in $\theta^{7/4}$ with a different coefficient. There is a clear disagreement between these three published results. The starting point of the present investigation was an attempt to decide which of them is correct.

The Bloch formula (1) is obtained by assuming that spin waves do not interact with one another, and that the energy of a spin wave is proportional to λ^{-2}, where λ is the wavelength. The theoretical deviations arise from three causes: (a) deviation of the energy spectrum from the λ^{-2} law; (b) true dynamical interaction between spin waves; (c) kinematical interaction between spin waves due to the fact that a single atom cannot carry more than $2S$ units of reversed spin simultane-

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