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On the Spin Wave Problem in the Heisenberg Model of Ferromagnetism

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The interacting spin system of the Heisenberg model is brought into contact with a system of free Bose particles which are independent of the spins. The partition function of the combined system differs from the original one only by a trivial factor. Without changing the thermodynamical properties, the total Hamiltonian H is transformed by means of nonsingular operators T according to THT^{-1}. The interaction between the spins is thereby eliminated to lowest order whereas the energy of the free Bose particles changes to the well-known energy dispersion of ideal spin waves. The interaction part of the transformed Hamiltonian describes ordinary scattering processes between two particles at least. Furthermore it allows at once for a low-temperature expansion in analogy to diagrammatical expansions of normal many particle systems. The expansion is studied in more detail. It is shown that contact can be made between this expansion and the low-temperature treatment of the Heisenberg model by Dyson. In particular, the lowest order contributions to the partition function may be calculated from a reduced Hamiltonian which coincides with Dyson's Hamiltonian for ideal spin waves with dynamical interaction. Therefore it is proved once again that Dyson's kinematical interaction is negligible at lowest temperatures.

I. Introduction

The low-temperature properties of the Heisenberg ferromagnet are well described by the concept of spin waves which were introduced into the problem by Bloch[1]. The spin waves are the excitations of lowest energy which determine the low-temperature expansion of the partition function or the free energy, respectively. In the limit of very low temperatures one may pass from physical spin waves to ideal spin waves which obey Bose statistics. Higher-order contributions to the free energy then result from the interaction between the ideal spin waves. This interaction consists of two parts; first we have the dynamical interaction contained in the Hamiltonian, second we have the kinematical interaction which results from the transition from physical to ideal spin waves. It is just this interaction which brings all the difficulties into the problem.

The solution of the problem was given by Dyson[2] in his monumental work on spin wave interactions. He showed that the effect of kinematical interactions is negligible at low temperatures, therefore we have to

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[1] Bloch, F.: Z. Physik **61**, 206 (1930); **74**, 295 (1932).

[2] Dyson, F. J.: Phys. Rev. **102**, 1217, 1230 (1956).

consider only the dynamical interaction which adds in lowest order a T^5-term to the free energy of free spin waves.

This paper deals with the same problem, but we shall use a method which is quite different from the one used by DYSON. We add a Hamiltonian of free Bose particles to the Heisenberg Hamiltonian. By a simple trick the two independent systems will be brought into contact with one another without changing the partition function of the combined system. A series of transformations will then bring the Hamiltonian in such a form that the interaction between spins is eliminated to lowest order whereas we obtain some interaction between the Bose particles. In particular the energy dispersion of the free Bose particles changes to the one of the well-known ideal spin waves. The total Hamiltonian describes an Ising model spin system in a spin wave bath. Furthermore the resulting interaction between the spin waves and the spins at once allows for a low-temperature expansion of the partition function in analogy to diagrammatical expansions of normal many-particle systems which obey Bose statistics. When we confine ourselves to the lowest order contributions, we regain DYSON's dynamical interaction.

We emphasize the fact that we need not introduce the ideal spin waves into HEISENBERG's Hamiltonian, therefore there is no kinematical interaction and we are not confronted with such difficulties as in DYSON's treatment.

II. The Heisenberg Model of Ferromagnetism

We consider a three-dimensional crystal of $\mathcal{N}$ lattice sites with periodic boundary conditions. To each lattice site j is attached a spin vector $\boldsymbol{\sigma}_j$ with the usual commutation relations,

$$[\sigma_i^x, \sigma_j^y] = \delta_{ij} \cdot i\, \sigma_j^z \quad \text{etc.}, \tag{1}$$

and the magnitude S which implies

$$\boldsymbol{\sigma}^2 = S(S+1). \tag{2}$$

The Hamiltonian is given by

$$H = -\varepsilon \sum_j \sigma_j^z - \tfrac{1}{2} \sum_{ij} V_{ij}\, \boldsymbol{\sigma}_i\, \boldsymbol{\sigma}_j \tag{3}$$

where the summations extend over all lattice sites j. The first term represents the magnetic energy of the spin system in an uniform magnetic field $\mathscr{H}$ along the z-axis with $\varepsilon = \frac{\mu_0 \mathscr{H}}{S}$, where μ_0 denotes the magnetic moment coupled to the spin. The exchange integral,

$$V_{ij} = V(|\boldsymbol{r}_i - \boldsymbol{r}_j|), \qquad V_{ii} = 0, \tag{4}$$

between spins at different lattice sites must be positive in order to describe a ferromagnetic behaviour of the system.

In the standard representation the σ^z-components may be chosen diagonal with the eigenvalue spectrum $-S, -S+1, \ldots, S$. As usual we introduce the operators

$$\left.\begin{aligned} n_j &= S - \sigma_j^z \\ \sigma_j^{\pm} &= \frac{1}{\sqrt{2S}}(\sigma_j^x \mp i\sigma_j^y) \end{aligned}\right\} \tag{5}$$

where n is diagonal with eigenvalues $0, 1, \ldots, 2S$. From (1) one deduces the commutation rules

$$\left.\begin{aligned} [n_i, \sigma_j^{\pm}] &= \pm\delta_{ij}\sigma_j^{\pm} \\ [\sigma_i^-, \sigma_j^+] &= \delta_{ij}\left(1 - \frac{n_j}{S}\right). \end{aligned}\right\} \tag{6}$$

In terms of the new operators the Hamiltonian is given by

$$H = E_0 + \alpha\sum_i n_i - \tfrac{1}{2}\sum_{ij} V_{ij}(n_i n_j + 2S\sigma_i^+\sigma_j^-) \tag{7}$$

with

$$E_0 = -\mathcal{N}(\varepsilon S + \tfrac{1}{2}S^2 v_0), \qquad \alpha = \varepsilon + S v_0. \tag{8}$$

E_0 is the groundstate energy and v_0 denotes the zeroth Fourier component of the potential

$$v_{\boldsymbol{p}} = \sum_j V(\boldsymbol{r}_j)\, e^{-i\boldsymbol{p}\boldsymbol{r}_j}; \tag{9}$$

the $\mathcal{N}$ possible $\boldsymbol{p}$ vectors all lie in the unit cell of the reciprocal lattice.

The operator n_i measures the component of the reversed spin at lattice site i, laxly speaking it is the number of spins (i.e. of reversed spins); similarly we may call the σ_i^+ and σ_i^- the creation and annihilation operators of a spin (i.e. of a reversed spin of magnitude $\frac{1}{2}$).

III. Introduction of Spin Waves

We now introduce a system of Bose particles independent of the spins by the definition of the Hamiltonian

$$H_B = \alpha\sum_i b_i^+ b_i, \tag{10}$$

i.e. to each lattice site we attach a creation and an annihilation operator b, b^+ with the commutation rules

$$[b_i, b_j^+] = \delta_{ij}. \tag{11}$$

The partition function of this system is simply given by

$$Z_B=(1-e^{-\beta\alpha})^{-\mathcal{N}}. \tag{12}$$

Combining the two systems, the partition function $Z=\operatorname{Trace}_{\sigma} e^{-\beta H}$ of the Heisenberg model may be written as

$$Z=e^{-\beta E_0}(1-e^{-\beta\alpha})^{\mathcal{N}}\cdot Z_1, \quad Z_1=\operatorname*{Trace}_{\sigma,b} e^{-\beta H_1} \tag{13}$$

where the trace* extends over all states of the total Hamiltonian

$$H_1=\alpha\sum_i(n_i+b_i^+ b_i)-\tfrac{1}{2}\sum_{ij}V_{ij}[n_i n_j+(\sigma_i^+ +b_i^+)\,\sigma_j^-\cdot 2S]. \tag{14}$$

This operator consists of the spin Hamiltonian (7), of the Bose particles Hamiltonian (10), and of the contact term $-S\sum_{ij}V_{ij}\,b_i^+\sigma_j^-$. The addition of the last term will not change the partition function as may be seen by calculating the trace on the Boson part.

Strictly speaking, the operator H_1 is no longer a Hamiltonian as it is obviously non-hermitian by the addition of the contact term. Furthermore the dynamical properties of the system described by the "Hamiltonian" H_1 do not coincide with those of the original system for the same reason. On the other hand we are interested only in the thermodynamics of the Heisenberg model, i.e. in the partition function Z; but this is equal to Z_1, up to a trivial factor, as may be seen from (13).

The Hamiltonian H_1 may now be transformed by a nonsingular operator T according to

$$H_1^T=T H_1 T^{-1} \tag{15}$$

without changing the partition function. First we choose

$$T_1=e^{-\sum_i \sigma_i^+ b_i} \tag{16}$$

and obtain with the help of the formulas (A.1) of the Appendix

$$\left.\begin{aligned}H_2=\alpha\sum_i(n_i+b_i^+ b_i)-\tfrac{1}{2}\sum_{ij}V_{ij}\Big\{&(n_i+\sigma_i^+ b_i)(n_j+\sigma_j^+ b_j)+\\ &+2S\,b_i^+\left[\sigma_j^- +\left(1-\frac{n_j}{S}\right)b_j-\frac{1}{2S}\,\sigma_j^+ b_j^2\right]\Big\}.\end{aligned}\right\} \tag{17}$$

This Hamiltonian splits up into a number of terms with different meaning. The pure spin part in (17) is equal to the Hamiltonian of the Ising model

$$H_{\text{Ising}}=\alpha\sum_i n_i-\tfrac{1}{2}\sum_{ij}V_{ij}\,n_i n_j, \tag{18}$$

* In the following the symbol *Trace* always means $\operatorname*{Trace}_{\sigma,b}$.

whereas the part which contains only Boson operators is given by

$$H_0 = \alpha \sum_i b_i^+ b_i - S \sum_{ij} V_{ij} b_i^+ b_j. \tag{19}$$

Fourier transformation of the Boson operators leads directly to

$$H_0 = \sum_p \varepsilon_p b_p^+ b_p \tag{20}$$

where

$$\varepsilon_p = \varepsilon + S(v_0 - v_p) \tag{21}$$

is the well-known energy of the ideal spin waves[2]. The rest of the Hamiltonian H_2 describes the interaction between the Ising spin system and the spin waves. We might say that the total system looks like an Ising spin system in a spin wave bath with particle exchange between spins and spin waves.

There is some analogy to the Bohm-Pines[3] treatment of electron-electron interactions. The collective coordinates of BOHM and PINES refer to the spin wave amplitudes which, as is well known, represent the collective excitations of the Heisenberg ferromagnet. However, there is an important difference. Whereas the number of degrees of freedom is not changed by the Bohm-Pines treatment, this number is enlarged enormously by the introduction of spin waves in the above treatment. Indeed, the spin wave operators are independent of the spin operators and the total Hilbert space is the Kronecker product of the spin space and the spin wave space.

The operator of the total number of particles is defined by

$$M = \sum_i (n_i + b_i^+ b_i) \tag{22}$$

and is a constant of motion, as it commutes both with H_1 and H_2. Therefore each subspace of the total Hilbert space, which is spanned by the statevectors $|\psi_M\rangle$ belonging to the same number M, is invariant with respect to H_2. If we apply the interaction terms in H_2 on the vectors $|\psi_M\rangle$, we see that there is only the term

$$-S \sum_{ij} V_{ij} b_i^+ \sigma_j^- \tag{23}$$

which may give a nonzero result when it acts on the vectors with $M=1$. All the other terms at least need $M=2$ in order to give nonzero results, i.e. these terms describe ordinary two-particle interactions.

We shall eliminate the term (23) which does not represent an ordinary interaction. This can be done by a further transformation according

[3] BOHM, D., and D. PINES: Phys. Rev. 92, 609 (1953).

to (15) with the nonsingular operator

$$T_2 = e^{\sum_i b_i^+ \sigma_i^-}$$

which commutes with M. Using the formulas (A.2) of the Appendix we obtain the following transformed Hamiltonian, $H_3 = T_2 H_2 T_2^{-1}$, which also commutes with M:

$$H_3 = H_0 + A - U. \tag{24}$$

The first term H_0 is the free spin wave part (20); the second term, A, contains only spin operators and replaces the Ising Hamiltonian (18),

$$A = \alpha \sum_i n_i - \frac{1}{8S^2} \sum_{ij} V_{ij}\, n_i(n_i-1)\, n_j(n_j-1), \tag{25}$$

whereas the third part U describes the interaction between spins and spin waves, and is given by

$$\left.\begin{aligned} U = \frac{1}{2} \sum_{ij} V_{ij} \Bigg\{ & (\sigma^+ b)_i \left[\sigma^+ b + \frac{1}{S} n(n-1)\right]_j + \\ & + \left[b^+ b - \frac{1}{S} b^+ \left(n + \frac{1}{2} b^+ \sigma^-\right)(b - \sigma^-)\right]_i \times \\ & \times \left[b^+ b - \frac{1}{S} b^+ \left(n + \frac{1}{2} b^+ \sigma^-\right)(b - \sigma^-) + 2\left(\sigma^+ b + \frac{1}{2S} n(n-1)\right)\right]_j - \\ & - b_i^+ \left[\left\{n + \frac{1}{2S} n(n-1) + \sigma^+ b + b^+(b + \sigma^-)\right\}(b - \sigma^-) - \right. \\ & \left. - \frac{1}{S} b^+ \left(n + \frac{1}{2} b^+ \sigma^-\right)(b - \sigma^-)^2\right]_j \Bigg\}. \end{aligned}\right\} \tag{26}$$

The individual terms in this expression are ordered in such a way that all creation operators σ^+ and b^+ stand to the left of the annihilation operators σ^- and b; clearly we only need to order the operators which refer to the same lattice site i since the coupling V_{ij} vanishes for $i=j$. Furthermore, in deducing (25) and (26) we have used the identity

$$n - \sigma^+ \sigma^- = \frac{1}{2S} n(n-1) \tag{27}$$

which follows from relation (2).

Though the total interaction U looks very complicated, it has the great advantage that the individual terms in (26) all describe ordinary interactions between two particles at least. We may classify the individual terms according to the minimal number $M_{\min}$ of particles we need in order that $U\,|\psi_{M_{\min}}\rangle$ gives a nonzero result. It is easy to see that in this sense U splits up into three sets of terms: $U(2)$, $U(3)$, and $U(4)$ which describe the scattering of two, three, and four particles, respectively. Furthermore, we see that the second part of A given by (25) describes a four-particle interaction between spins.

Let us consider for a moment the operator A. In the case of spin $\frac{1}{2}$ $(S=\frac{1}{2})$, the second part vanishes because n can take only the eigenvalues 0 and 1; A therefore reduces to the one-particle operator $\alpha_0 \sum_i n_i$ which may be denoted by A_0. It is easy to show that A is positiv-semidefinite and the eigenvalues of A_0 are lower bounds of the corresponding eigenvalues of A: Using $n(n-1)\leqq 4S^2$, we obtain

$$A \geqq \sum_i [\varepsilon n_i + v_0(S n_i - \tfrac{1}{2} n_i^2 + \tfrac{1}{2} n_i)] \geqq \alpha_0 \sum_i n_i = A_0 \quad (\alpha_0 = \varepsilon + \tfrac{1}{2} v_0). \tag{28}$$

We shall need this useful property in the next section, where we study the low-temperature behaviour of the Heisenberg ferromagnet.

IV. Low-Temperature Expansion of the Partition Function

The Hamiltonian H_3 will now be used to obtain a low-temperature expansion of the partition function $Z' = \text{Trace}\, e^{-\beta H_3}$. If we take the interaction part as perturbation, we can disentangle the unperturbed part, $H_0 + A$, from the exponential and the partition function may be written as

$$Z' = \text{Trace}\, e^{-\beta(H_0+A)}\, T \exp\left\{\int_0^1 dt\, \beta\, U(t)\right\} \tag{29}$$

where t is FEYNMAN's[4] ordering parameter or, as one may say, a dimensionless imaginary time, and T denotes the usual time ordering symbol. In the interaction representation we have

$$U(t) = e^{t\beta(H_0+A)}\, U\, e^{-t\beta(H_0+A)}. \tag{30}$$

Expanding (29) with respect to U, we obtain

$$Z' = \text{Trace}\, e^{-\beta(H_0+A)} \left\{1 + \sum_{\nu=1}^{\infty} \int_0^1 dt_\nu \int_0^{t_\nu} dt_{\nu-1} \ldots \int_0^{t_2} dt_1\, \beta\, U(t_\nu) \ldots \beta\, U(t_1)\right\} \tag{31}$$

which is ordered by the succession of the integrations.

Starting from this expression one could try to define an expansion in diagrams. But a diagrammatical expansion is not simple if one deals with spin operators, since one cannot apply WICK's theorem[5] generalized to thermodynamical averages[6]. Instead of the usual treatment one has

[4] FEYNMAN R. P.: Phys. Rev. **84**, 108 (1951).

[5] WICK, G.: Phys. Rev. **80**, 268 (1950).

[6] BLOCH, C., and C. DE DOMINICIS: Nuclear Phys. **7**, 459 (1958).

to consider a semiinvariant expansion, as was done for the Heisenberg Hamiltonian by STINCHCOMBE et al.[7]. In the following we need not define diagrams, but we emphasize the fact that the "linked-cluster-theorem" at once follows from a semiinvariant expansion, therefore the free energy, $\beta F' = -\log Z'$, only contains contributions from connected diagrams which are proportional to the number of lattice sites $\mathcal{N}$.

In this chapter we confine our attention to the temperature dependence of the individual terms in the expansion (31). Temperature-dependent factors may come from two sources. First, there are the β-factors connected with each U-term in (31); these β-factors are partially compensated by the time integrations as is shown below. Second, several exponential factors will appear which result from the interaction representation of the U-terms.

As usually we shall neglect contributions to the partition function or the free energy, respectively, which decrease exponentially with temperature like $e^{-\beta x}$ $(x>0)$. Introducing the explicitly time-dependent $U(t)$ given by (30) into the expansion (31), and taking into account that the unperturbed part, H_0+A, is positiv-semidefinite, we see at once that, after integration, the exponential temperature dependence of a special term in the expansion will be given by $e^{-\beta x}$, where the number x is positive or zero $(x\geqq 0)$. Furthermore, if the same contribution is calculated with A_0 replacing A, we conclude from (28) that the corresponding number x_0 satisfies the condition

$$0\leqq x_0\leqq x\,. \tag{32}$$

Therefore we may use A_0 instead of A in order to determine the temperature dependence of a matrix element in the expansion (31), because the error can be only of the order $e^{-\beta(x-x_0)}$.

Now it is easy to calculate the right hand side of (30). The operators in U are transformed according to

$$e^{t\beta(H_0+A_0)}\begin{Bmatrix} b_i^+ \\ b_i \\ \sigma_i^\pm \\ n_i \end{Bmatrix} e^{-t\beta(H_0+A_0)} = \begin{Bmatrix} \frac{1}{\sqrt{\mathcal{N}}}\sum_p \begin{matrix} b_p^+ \\ b_p \end{matrix}\, e^{\pm i p r_i}\cdot e^{\pm\beta\varepsilon_p t} \\ \sigma_i^\pm\, e^{\pm\beta\alpha_0 t} \\ n_i \end{Bmatrix}. \tag{33}$$

The first term in (31) gives, up to exponentially small contributions, the partition function of ideal spin waves:

$$Z_0 = \operatorname*{Trace}_b e^{-\beta H_0} = \prod_p \{1-e^{-\beta\varepsilon_p}\}^{-1}. \tag{34}$$

[7] STINCHCOMBE, R.B., G. HORWITZ, F. ENGLERT, and R. BROUT: Phys. Rev. **130**, 155 (1963)

If we define the average value of an operator $\mathcal{O}$ by

$$\langle \mathcal{O} \rangle = \frac{\text{Trace}\, e^{-\beta H_0} \mathcal{O}}{\text{Trace}\, e^{-\beta H_0}}, \tag{35}$$

the partition function Z''* may be written as

$$\left.\begin{aligned} \frac{Z''-Z_0}{Z_0} = \sum_{\nu=1}^{\infty} \int_0^1 d t_\nu \ldots \int_0^{t_2} d t_1 \times \\ \times \sum_{N=0}^{\infty} \sum_{l} \langle \varphi_l(N) |\, e^{-\beta A_0} \langle \beta\, U(t_\nu) \ldots \beta\, U(t_1) \rangle \,| \varphi_l(N) \rangle \end{aligned}\right\} \tag{36}$$

where the l-summation extends over all orthonormalized state-vectors $|\varphi_l(N)\rangle$ belonging to the same number of spins $N=\sum_i n_i$**. The Bose average may be performed using Wick's theorem, and is given by the sum of all possible pair-contractions defined by***

$$\langle T(b_{\boldsymbol{p}}(t)\, b^+_{\boldsymbol{p}'}(t')) \rangle = \delta_{\boldsymbol{p}\,\boldsymbol{p}'} \left[\Theta(t-t') + \frac{1}{e^{\beta\,\varepsilon_p}-1} \right] e^{-\beta\,\varepsilon_p (t-t')}; \tag{37}$$

in particular, if the exponential factor $e^{-\beta\,\varepsilon_p (t-t')}$ vanishes by integrations and $t'>t$, relation (37) leads to the "particle" average

$$\langle b^+_{\boldsymbol{p}}\, b_{\boldsymbol{p}} \rangle = (e^{\beta\,\varepsilon_p}-1)^{-1}. \tag{38}$$

Introducing the spin wave energy (21) with $(v_0 - v_p) \approx \gamma\, \boldsymbol{p}^2$, we see that in zero magnetic field the average will decrease exponentially if the momentum $\boldsymbol{p}$ is not small as $\sqrt{T}$, whereupon the $\boldsymbol{p}$-summation will give a $T^{\frac{3}{2}}$-factor.

Let us now show that the time-integrations in (36) compensate the β-factors connected with the U-terms. We consider the first integration over t_1. The β in front of $U(t_1)$ will not be compensated if the argument of the t_1-dependent exponential function, due to the interaction representation, vanishes identically. From Wick's theorem we conclude that this may occur only if the corresponding b^+ and b are contracted pairwise, i.e. if we have the Bose average $\langle U(t_1) \rangle$. Certainly this average will not vanish if U contains an equal number of b^+ and b. However, we see from (26) that the U-terms in question all have at least one b^+

* The double prime refers to A_0.

** The state-vectors $|\varphi_l(N)\rangle$ span the total spin Hilbert space $\left(\sum_{lN} |\varphi_l(N)\rangle\langle\varphi_l(N)| = 1\right)$ and may be combined from the vectors $|n_i\rangle$ belonging to each lattice site i.

*** There is no ambiguity concerning the Θ-function if the arguments t and t' are equal because the b^+ always stands to the left of b in each U-term. Therefore the time argument of b^+ is greater by an infinitesimal and the Θ-function vanishes.

in front; therefore we obtain at least one particle average (38) which gives a $T^{\frac{3}{4}}$-factor compensating the β-factor.

Suppose now that after μ integrations the argument of the exponential function depending on $t_{\mu+1}$ vanishes. According to WICK's theorem we then have to consider the Bose average of some U, i.e. $\langle U \ldots U \rangle$. There might be no particle contraction (38) at all, but only "hole" contractions $\langle b\, b^+ \rangle$, and therefore no compensating $T^{\frac{3}{4}}$-factor. However, this possibility will occur only if the U-term in front of the average is given by one of the first U-terms in (26):

$$U_1 = \frac{1}{2} \sum_{ij} V_{ij} (\sigma^+ b)_i \left[\sigma^+ b + \frac{1}{S} n(n-1) \right]_j . \tag{39}$$

Let us consider the contribution of the second term in more detail*. Combining in formula (36) the critical integration over $t_{\mu+1}$ with the preceding integration over t_μ, we have, neglecting all parts we are not interested in,

$$\left. \begin{aligned} &\beta^2 \int_0^{t_{\mu+2}} d t_{\mu+1}\, e^{t_{\mu+1} \beta (\alpha_0 - \varepsilon_p)} \cdot \int_0^{t_{\mu+1}} d t_\mu\, e^{-t_\mu \beta (\alpha_0 - \varepsilon_p)} \\ &= (\alpha_0 - \varepsilon_p)^{-2} \left[e^{t_{\mu+2} \beta (\alpha_0 - \varepsilon_p)} - 1 + (\alpha_0 - \varepsilon_p) \left(\frac{\partial}{\partial \varepsilon_p} e^{t_{\mu+2} \beta (\alpha_0 - \varepsilon_p)} \right)_{\alpha_0 = \varepsilon_p} \right] \end{aligned} \right\} \tag{40}$$

where we have introduced the time-dependent factors of the operators $\sigma_i^+(t_{\mu+1})$ and $b_i(t_{\mu+1})$ (33). The third term in (40) is the critical one. Performing now the other integrations in (36) with the exponential $e^{t_{\mu+2} \beta (\alpha_0 - \varepsilon_p)}$, we may generate the critical contribution by applying the operation $\frac{\partial}{\partial \varepsilon_p} \cdots \big|_{\alpha_0 = \varepsilon_p}$ on the final contribution of the first noncritical term in (40). The result of all integrations will be a sum of exponential factors with denominators of the form

$$\frac{e^{\beta (a_1 + \alpha_0 - \varepsilon_p)}}{(a_2 + \alpha_0 - \varepsilon_p)(a_3 + \alpha_0 - \varepsilon_p) \ldots}, \qquad \frac{e^{\beta b_1}}{(b_2 + \alpha_0 - \varepsilon_p)(b_3 + \alpha_0 - \varepsilon_p) \ldots}.$$

Applying the above-mentioned operation on these terms, we shall obtain the critical β-factor only if the operation is applied to the exponential factor $e^{\beta (a_1 + \alpha_0 - \varepsilon_p)}$. Combining the final contributions of the critical and the noncritical term in (40), we have

$$e^{\beta (a_1 + \alpha_0)} \left[e^{-\beta \varepsilon_p} - \beta (\alpha_0 - \varepsilon_p)\, e^{-\beta \alpha_0} \right].$$

The first term in brackets will give a $T^{\frac{3}{4}}$-contribution if $\boldsymbol{p}$ is small as $\sqrt{T}$, whereas the second critical contribution decreases exponentially and may be neglected.

* The following discussion is similar for the contribution of the first term in (39).

Thus we obtain the result that the β-factors of a special matrix element in (36) are compensated by the integrations or by Bose averages if in intermediate integrals the arguments of the time-dependent exponentials vanish. As we are interested only in the lowest order contributions with respect to β, we can neglect the last case as for the further $\sqrt{T}$-dependence introduced by the Bose average $\langle b^+ b\rangle$. There may be only one β-factor if the argument of the exponential function in the last integral, $\int_0^1 dt_\nu \ldots$, vanishes.

Let us now consider a special spin matrix element $\langle\varphi_l(N)|\ldots|\varphi_l(N)\rangle$ in (36). As the spin operators are independent for different lattice sites, it is sufficient to consider the contribution of a single lattice point i:

$$e^{-\beta\alpha_0 n}\langle n|\,T\big(\sigma^+(t_1)\ldots\sigma^+(t_\mu)\,\sigma^-(t_1')\ldots\sigma^-(t_\mu')\big)\,|n\rangle \tag{41}$$

where we have omitted n-operators which have no influence on the temperature dependence. Furthermore we can assume that all the time arguments of the σ^+ are larger than those of the σ^-; otherwise there would be some $\sigma^-(t')\,\sigma^+(t)$ $(t'>t)$ which might be replaced by the number $\sigma^-\sigma^+$ enlarging the corresponding exponential $e^{-\beta\alpha_0(t'-t)}$ to unity. Introducing $\sigma^\pm(t)=\sigma^\pm\,e^{\pm\beta\alpha_0 t}$, it is easy to see that (41) vanishes identically for $\mu>n$, exponentially for $\mu<n$, and that it gives a finite result only if $\mu=n$ and $t_1=\cdots t_\mu=1$, $t_1'=\cdots t_\mu'=0$. Therefore we conclude that a special matrix element $\langle\varphi_l(N)|\ldots|\varphi_l(N)\rangle$ will give a finite result, only if we can find N σ^+-operators* with larger time arguments than those of the corresponding σ^--operators; furthermore, only the upper (lower) limits of the integrations over the time arguments of the $\sigma^+(\sigma^-)$ will contribute.

As may be seen from (26), all U-terms which contain a σ^+ just have one b-operator more than b^+-operators. Therefore, the N σ^+-operators are always accompanied by a factor $e^{-\beta(\varepsilon_{q_1}+\cdots+\varepsilon_{q_N})}$ which comes from the time-dependent exponentials $e^{-\beta\varepsilon_q t}$ connected with each b_q. Moreover, the integrations in (36) can give only an equal number ν of $e^{\beta\varepsilon_p}$- and $e^{-\beta\varepsilon_{q'}}$-factors. Altogether we have the exponential factor

$$e^{\beta(\varepsilon_{p_1}+\cdots+\varepsilon_{p_\nu}-\varepsilon_{q_1'}-\cdots-\varepsilon_{q_\nu'}-\varepsilon_{q_1}-\cdots-\varepsilon_{q_N})} \tag{42}$$

with independent momenta $p_1\ldots q_N$**. As all the b_p^+-operators ($p=p_1\ldots p_\nu$) stand in front of their contraction partners b_p — otherwise it would not be possible to obtain $e^{\beta\varepsilon_p}$ from the corresponding time-

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* In particular, if $N=n_{i_1}+n_{i_2}+\cdots$, we must have $n_{i_1}\,\sigma_{i_1}^+$, $n_{i_2}\,\sigma_{i_2}^+$, etc.

** Pair-contractions between some $b_p^+(p=p_1\ldots p_\nu)$ and some $b_q(q=q_1'\ldots q_\nu' q_1\ldots q_N)$ will merely decrease the number ν.

dependent factor $e^{\beta\varepsilon_p(t-t')}$ by integration –, we have for $p=p_1 \ldots p_\nu$

$$e^{\beta\varepsilon_p}\langle b_p^+ b_p\rangle = \langle b_p b_p^+\rangle .$$

Similarly the b_q ($q=q_1 \ldots q_N$, $q'_1 \ldots q'_\nu$) will give

$$e^{-\beta\varepsilon_q}\langle b_q b_q^+\rangle = \langle b_q^+ b_q\rangle .$$

As all "particle"-contractions lead to a $T^{\frac{3}{2}}$-factor, we obtain the result that a special matrix element in (36) with N reversed spins and F "particle" contractions $\langle b^+ b\rangle$ at least must give the total temperature-factor

$$T^{\frac{3}{2}\cdot(F+N)-1}. \tag{43}$$

It may be noticed that $F \geqq \nu$ because there may be several other contractions $\langle b^+ b\rangle$ in the matrix element.

The number $(F+N)$ is just the total number of independent particles which interact with one another; N refers to the number of reversed spins in the entrance channel, whereas F is the number of momenta independent spin wave "particles".

V. Lowest-Order Contribution to the Partition Function

Let us consider the different interaction terms U in (26). As the operators b and b^+ contained in the U-terms always refer to independent spin wave momenta, we see that a special $U(M_{\min})$ describes the interaction of $M_{\min}$ independent particles. Therefore, if a matrixelement in (36) contains a special term $U(M_{\min})$, the total number $(F+N)$ of independent particles cannot be smaller than $M_{\min}$, i.e.

$$F+N \geqq M_{\min} . \tag{44}$$

In order to obtain the lowest order contributions we can therefore neglect the terms $U(3)$ and $U(4)$ because these will lead to temperature-factors $T^{7/2}$ or T^5, respectively, at least. Furthermore we may neglect the second part of the operator A (25) for the same reason. Thus, the lowest order contributions to the partition function may be calculated from the reduced Hamiltonian H_3 (24),

$$\left.\begin{aligned} H_3^{\text{red}} &= \alpha\sum_i (n_i + b_i^+ b_i) - \\ &-\tfrac{1}{2}\sum_{ij} V_{ij}\Big\{\sigma_i^+ \sigma_j^+ b_i b_j + b_i^+ b_j^+ b_i b_j + 2 b_i^+ \sigma_j^+ b_i b_j + \\ &+ 2S\, b_i^+ \Big[b + \underline{\frac{1}{2S} n\sigma^-} + \underline{\frac{1}{2S} b^+ \sigma^{-2}} - \frac{1}{2S}\sigma^+ b^2 - \frac{1}{2S} b^+ b^2 \Big]_j \Big\} \end{aligned}\right\} \tag{45}$$

where we have retained only the two-particle interactions.

By further transformations with nonsingular operators T, according to (15), we shall now eliminate the underlined interaction terms in (45) which actually contribute to higher order. Using*

$$T_3 = e^{\frac{1}{2S}\sum_i b_i^+ n_i \sigma_i^-}, \tag{46}$$

we can eliminate the term

$$-\tfrac{1}{2}\sum_{ij} V_{ij}\, b_i^+ n_j \sigma_j^- \tag{47}$$

by means of (A.3). Instead of this term we obtain a number of interaction terms which describe the scattering of more than two particles. According to the analysis of the last chapter we omit these terms in the Hamiltonian. In the same way we eliminate the term

$$-\tfrac{1}{2}\sum_{ij} V_{ij}\, b_i^+ b_j^+ \sigma_j^{-2} \tag{48}$$

from the Hamiltonian by the transformation with

$$T_4 = e^{\frac{1}{4S}\sum_i b_i^{+2} \sigma_i^{-2}}, \tag{49}$$

neglecting all interactions between three and more particles.

Now it is easy to see that we can neglect all other terms in (45) which contain at least one spin operator. These interaction terms do not contribute to the partition function, because the σ^+-operators cannot lead to diagonal matrixelements. Furthermore

$$\operatorname*{Trace}_{\sigma} e^{-\beta\alpha\sum_i n_i} = 1 + O(e^{-\beta\alpha}). \tag{50}$$

Thus in lowest order beyond the contribution of free spin waves, the partition function can be calculated from the resulting Hamiltonian

$$H_{\text{Dyson}} = \sum_i \alpha\, b_i^+ b_i - S\sum_{ij} V_{ij}\, b_i^+ b_j + \tfrac{1}{2}\sum_{ij} V_{ij}\, b_i^+ b_j^+ b_j (b_j - b_i), \tag{51}$$

which may be Fourier-transformed to

$$\left.\begin{aligned} H_{\text{Dyson}} = {} & \sum_p \varepsilon_p\, b_p^+ b_p + \\ & + \frac{1}{2\mathcal{N}} \sum_{p\,p'\,q\,q'} \delta(p+q-p'-q')(v_{p'} - v_{q-p'})\, b_{p'}^+ b_{q'}^+ b_p b_q . \end{aligned}\right\} \tag{52}$$

This is DYSON's[2] Hamiltonian, where the second part is just the dynamical interaction between ideal spin waves.

* The first part of H_3^{red} will not change since T_3 commutes with M.

From the analysis of the last section we conclude that the lowest order contribution to the partition function should be given by the scattering of two spin waves with a temperature factor T^2. Actually, as may be seen from DYSON's work, the final contribution is of the order T^4. This difference is due to the symmetry of the interaction potential.

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Appendix

The following formulas may be checked by using the commutation rules (6) and (11) and differentiation with respect to λ; all the operators refer to the same lattice site j, which is omitted for simplicity:

$$\left.\begin{aligned} T_1 = e^{-\lambda \sum_i \sigma_i^+ b_i} : e^{-\lambda \sigma^+ b} &\begin{Bmatrix} b \\ b^+ \\ n \\ \sigma^- \\ \sigma^+ \end{Bmatrix} e^{\lambda \sigma^+ b} \\ &= \begin{Bmatrix} b \\ b^+ - \lambda \sigma^+ \\ n + \lambda \sigma^+ b \\ \sigma^- + \lambda \left(1 - \frac{n}{S}\right) b - \frac{\lambda^2}{2S} \sigma^+ b^2 \\ \sigma^+ \end{Bmatrix}, \end{aligned}\right\} \tag{A.1}$$

$$\left.\begin{aligned} T_2 = e^{\lambda \sum_i b_i^+ \sigma_i^-} : e^{\lambda b^+ \sigma^-} &\begin{Bmatrix} b \\ b^+ \\ n \\ \sigma^- \\ \sigma^+ \end{Bmatrix} e^{-\lambda b^+ \sigma^-} \\ &= \begin{Bmatrix} b - \lambda \sigma^- \\ b^+ \\ n + \lambda b^+ \sigma^- \\ \sigma^- \\ \sigma^+ + \lambda b^+ \left(1 - \frac{n}{S}\right) - \frac{\lambda^2}{2S} b^{+2} \sigma^- \end{Bmatrix}, \end{aligned}\right\} \tag{A.2}$$

$$
\left.\begin{aligned}
T_3 &= e^{\lambda \sum_i b_i^+ n_i \sigma_i^-} : e^{\lambda b^+ n \sigma^-} \begin{Bmatrix} b \\ b^+ \\ n \\ \sigma^- \\ \sigma^+ \end{Bmatrix} e^{-\lambda b^+ n \sigma^-} \\
&= \begin{Bmatrix} b - \lambda n \sigma^- \\ b^+ \\ n + \lambda b^+ n \sigma^- \\ \sigma^- [1 + \lambda b^+ \sigma^-]^{-1} \\ \left[\sigma^+ + \lambda b^+ \left(1 - \frac{n + \frac{1}{2}\lambda b^+ n \sigma^-}{S}\right) n\right] [1 + \lambda b^+ \sigma^-] \end{Bmatrix},
\end{aligned}\right\} \quad \text{(A.3)}
$$

$$
\left.\begin{aligned}
T_4 &= e^{\lambda \sum_i b_i^{+2} \sigma_i^{-2}} : e^{\lambda b^{+2} \sigma^{-2}} \begin{Bmatrix} b \\ b^+ \\ n \\ \sigma^- \\ \sigma^+ \end{Bmatrix} e^{-\lambda b^{+2} \sigma^{-2}} \\
&= \begin{Bmatrix} b - 2\lambda b^+ \sigma^{-2} \\ b^+ \\ n + 2\lambda b^{+2} \sigma^{-2} \\ \sigma^- \\ \sigma^+ + \lambda b^{+2} \left[2 - \frac{1}{S}(2n + 2\lambda b^{+2} \sigma^{-2} + 1)\right] \sigma^- \end{Bmatrix}.
\end{aligned}\right\} \quad \text{(A.4)}
$$