Physics 216: Topics in many-body theory, spring 2016

Problem set 1: assigned 1/26/16, due 2/11/2016 end of class or 4 pm my office door, based on lectures 1-6

1. A warmup on variational methods: (similar to Ashcroft and Mermin problems 17.1 and 17.2) Show that Hartree-Fock equations like those given in lecture I are obtained by variational minimization over N-electron Slater determinants of the Hamiltonian

$$H_e = \sum_{i=1}^{N} \left(-\frac{\hbar^2}{2m} \nabla_i^2 + V_b(r_i) \right) + \frac{1}{2} \sum_{i \neq j} \frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}. \tag{1}$$

It may help to show first that the Hartree equations come from minimization over product wavefunctions (not antisymmetrized).

2. Show that, in BCS theory, it does not modify the fluctuations in the number operator while preserving $|u_k|^2$ and $|v_k|^2$ by introducing phases into u_k and v_k. Can you find a different way to minimize the fluctuations?

Hint: define

$$|\Psi_{BCS}(\theta)\rangle = \prod_k (u_k + v_k e^{i\theta} b_k^\dagger)|0\rangle \tag{2}$$

and integrate

$$\int_0^{2\pi} e^{-iN\theta} |\Psi_{BCS}(\theta)\rangle \, d\theta \tag{3}$$

where here N is the desired number of Cooper pairs. How does the energy of this state compare to the original state with u_k and v_k real? How does the number compare?

3. (a) Write out the details for yourself of the argument that E_k is the energy increase above the ground state of a broken Cooper pair at k; (b) Connect the gap parameter Δ in the BCS wavefunction to the idea of "off-diagonal long-range order" as follows: write the operator $c_\uparrow^\dagger c_\downarrow^\dagger$ at a point in space as an integral over operators in momentum space (i.e., $c_{\sigma k}^\dagger$ operators), then compute the expectation value of this operator and show that it is proportional to $u_k v_k$, the same combination that also appears in Δ.

4. How much is the kinetic energy increased in the BCS state compared to the Fermi sea, for the simple solvable model

$$V_{kk'} = \begin{cases} -V < 0 & \text{if } |\epsilon_k - \mu| \text{ and } |\epsilon_{k'} - \mu| < \omega_c \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

How much is the interaction energy reduced?

5. Obtain the compressibility formula in Fermi liquid theory

$$\frac{\partial P}{\partial \rho} = \frac{{p_F}^2}{3mm^*}(1 + F_0) = \frac{{p_F}^2}{3m^2} \frac{1 + F_0}{1 + F_1}. \tag{5}$$

You will probably want to start from the relation

$$\frac{\partial \mu}{\partial N} = -\frac{V^2}{N^2} \frac{\partial P}{\partial V} \tag{6}$$

so that the compressibility is just $(N/m)\frac{\partial\mu}{\partial N}$. The next step is to write (you should justify this)

$$\delta\mu = \int f(\mathbf{p}_F, \mathbf{p}')\,\delta n'\,\delta\tau' + \frac{\partial\epsilon_F}{\partial p_F}\,\delta p_F. \tag{7}$$

Feel free to consult chapter 2 of Landau and Lifshitz volume 9 if you get stuck.

6. Quick (I hope) numbers problem: Suppose a superconductor is well described by s-wave BCS theory and has a transition temperature $T_c = 10K$. Use the relationship between T_c and peak Δ to estimate the maximum of Δ. If the Fermi velocity is 10^5 meters per second, what is the coherence length ξ? If the electronic density is that of aluminum, how many electrons are there inside a sphere of radius ξ? Read about the AC Josephson effect if you haven't encountered it before. What is the Josephson frequency in Hertz for a voltage of one volt?

On your own time (i.e., no need to write it up): I mentioned the Anderson-Higgs mechanism in class. There is a pretty good Wikipedia article `http://en.wikipedia.org/wiki/Higgs_mechanism` that uses our non-relativistic superconductor case as an example. If you're interested you might enjoy reading it and understanding how the fact that the condensate in a superconductor is coupled to the electromagnetic gauge field leads to a gap of the "Goldstone boson" (the gapless mode that normally results when a continuous symmetry is broken). In a superfluid, where the condensate is neutral, there is a gapless mode, while in the superconductor, the oscillation ("Higgs mode", recently observed directly) gets pushed up to the plasma frequency.