Physics 216: Topics in many-body theory, spring 2016

Problem set 2: assigned 2/15/16, due 2/25/16

1. Use spin-wave theory to show, first, that even at zero temperature there is no long-range order in 1D (as sketched in class), and second, to estimate how much the antiferromagnet moment is reduced by quantum fluctuations in 2D and 3D, for the cubic lattice, for $S = \frac{1}{2}$. Auerbach will be helpful.

The known answer from Quantum Monte Carlo simulations in 2D is that for spin-half the moment is reduced by about 40 percent from its classical value, which agrees well with neutron scattering measurements.

2. Consider a Cooper pair in He^3, which is a p-wave superconductor, so the angular momentum of the pair is $\hbar$. Estimate the order of magnitude of the Cooper pair size as follows: assume that the gap maximum is of the same order as T_c (of order 10^{-3} K). Assume that the Cooper pair size, of order ξ, is related to the gap through

$$\xi \sim \frac{\hbar v_F}{\Delta}. \tag{1}$$

What is the order of magnitude of the rotational velocity of one He atom in a pair? You may wish to use the fact that the effective mass is $m^* = 3.1m$.

Compare the rotational velocity to the Fermi velocity, which you can estimate by recalling that the effective mass $m^* = 3.1m$ and knowing that the Fermi momentum is $p_F = \hbar(0.8 \times 10^8 \text{ cm}^{-1})$.

3. Consider the following tight-binding-like model (one variant of the Hubbard model) on the square lattice with one orbital per site:

$$H = -t \sum_{\langle ij \rangle, \sigma} (c_{i\sigma} c^\dagger_{j\sigma} + h.c.) - \sum_i U n_{i\uparrow} n_{i\downarrow} \tag{2}$$

The first sum is over nearest-neighbor pairs.

Suppose that U is positive (so that there is an attractive interaction whenever two particles are on the same site) and weak compared to the hopping t. Also suppose that the system is at half-filling (one electron per site on average). Start by solving the $U = 0$ problem. Then calculate the $V_{kk'}$ matrix elements induced by U that should appear in the pairing Hamiltonian as functions of t. What do you expect to be the angular momentum of the Cooper pairs? (i.e., s-wave, p-wave, etc.) Estimate the transition temperature in terms of the original parameters of the model.

4. Consider a triangle of spin-half spins coupled antiferromagnetically, i.e.,

$$H = J\left[(\mathbf{s}_1 \cdot \mathbf{s}_2) + (\mathbf{s}_2 \cdot \mathbf{s}_3) + (\mathbf{s}_1 \cdot \mathbf{s}_3)\right]. \tag{3}$$

Here $J > 0$ and each of the 3 spins is coupled to the other 2. (a) Find the spectrum of this Hamiltonian, which should have 8 states. (b) What is the mean energy for one of the terms in the Hamiltonian in the ground state? How does this compare to the energy of a singlet?