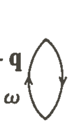
Diagrams of form (1), (2), (3), ..., having one interaction line entering and one leaving are called '*polarization diagrams*'. The reason for this is that they show how the interaction causes the medium to become 'virtually polarized' in all possible ways. For example, if we regard diagram (2) in (10.34) as drawn in $(\mathbf{r}, t)$-space, with the lower vertex of the 'pair bubble' at $\mathbf{r}_1, t_1$ and the upper at $\mathbf{r}_2, t_2$, then for $t_1 < t < t_2$ there are a negative electron and a positive hole separated in space forming a 'virtual dipole'. (Of course, the position co-ordinates of the particle and hole are no longer sharp for $t > t_1$ but this makes no difference—the virtual dipole just becomes 'fuzzy'.)

Equation (10.34) may be written in functional form as

$$V_{\text{eff(RPA)}}(\mathbf{q}, \omega) = \frac{V_q}{1 + V_q \pi_0(\mathbf{q}, \omega)} \equiv \frac{V_q}{\epsilon_{\text{RPA}}(\mathbf{q}, \omega)} \tag{10.35}$$

where

$$-i\pi_0(\mathbf{q}, \omega) \equiv$$



(10.36)

This has the form of an interaction taking place between two charges in a dielectric, with

$$\epsilon_{\text{RPA}}(\mathbf{q}, \omega) = 1 + V_q \pi_0(\mathbf{q}, \omega) \tag{10.37}$$

being the frequency-dependent or so-called 'generalized' dielectric constant. Of course, this is no coincidence. The dielectric properties of a medium arise just because of the polarization of the medium by a field, and (10.34) is just the sum of diagrams representing the polarization of the electron gas by the field of one of the electrons in the gas itself. Note that $V_{\text{eff}}(\mathbf{q}, \omega)$ depends on ω, unlike the bare V_q. If V_{eff} is Fourier transformed to $(\mathbf{q}, t)$-space, it will thus be a time-dependent interaction; this is due to the inertia of the polarization charge.

We may evaluate $\pi_0(\mathbf{q}, \omega)$ by using the rules for graphs, Table 9.1, yielding

$$-i\pi_0(\mathbf{q}, \omega) = 2 \times (-1) \int \frac{d^3\mathbf{k}\, d\epsilon}{(2\pi)^4} \frac{i}{\omega + \epsilon - \epsilon_{k+q} + i\delta_{k+q}} \times \frac{i}{\epsilon - \epsilon_k + i\delta_k}, \tag{10.38}$$

where the factor of 2 comes from the sum over spins and the (-1) from the fermion loop. The integral over ϵ is in (9.55–62) and $\int d^3\mathbf{k}$ is done in §10.7. In the limit when $\omega = 0$ and $\mathbf{q}$ is small, it is found that

$$\pi_0(q \ll k_F, \omega = 0) = \frac{\lambda^2}{4\pi e^2}, \quad \lambda^2 = \frac{6\pi n e^2}{\epsilon_F} = \frac{4me^2 k_F}{\pi} = \left(\frac{2}{\pi}\right)\left(\frac{4}{9\pi}\right)^{1/3} r_s k_F^2 \tag{10.39}$$

where n = electron density $= \frac{1}{3}\pi^{-2} k_F^3$, $\epsilon_F = k_F^2/2m$.

It is now possible to calculate $V_{\text{eff}}(\mathbf{q}\,\text{small}, 0)$. Setting $\Omega = 1$ and omitting spins for simplicity in (7.71), we have

$$V_q = \frac{4\pi e^2}{q^2}. \tag{10.40}$$

Substituting this together with (10.39) into (10.35) yields

$$V_{\text{eff(RPA)}}(\text{small } \mathbf{q}, 0) = \frac{4\pi e^2}{q^2 + \lambda^2} \tag{10.41}$$

(valid in the limit $(\lambda/k_F)^2 \ll 1$, i.e., $r_s \ll 1$. See 10.82). Hence, assuming (10.41) holds for all q,

$$V_{\text{eff}}(r) \propto \frac{e^2}{r} e^{-\lambda r} \tag{10.42}$$

which is a shielded Coulomb interaction. (See (10.83), for a more correct expression!) This reveals the physical significance of the effective interaction: the bare interaction (10.40) virtually polarizes the medium, and the polarization cloud in turn shields the bare interaction converting it into the much weaker effective interaction. (This effective interaction turns out to be just the effective interaction between quasi particles (see Fig. 0.12), as discussed in Falicov (1961).)

We can now go on to the evaluation of $\Sigma_{\text{RPA}}(\mathbf{k}, \omega)$ as it appears in (10.33). Translating this into functions with the aid of (10.35) gives

$$-i \sum_{\text{RPA}} (\mathbf{k}, \omega) = \sum_{\mathbf{q}} \int \frac{d\gamma}{2\pi} [(-i)\, V_{\text{eff(RPA)}}(\mathbf{q}, \omega)]\, [iG_0(\mathbf{k} - \mathbf{q}, \omega - \gamma)]$$

$$= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \int \frac{d\gamma}{2\pi} \frac{4\pi e^2}{q^2 \epsilon_{\text{RPA}}(\mathbf{q}, \gamma)} \times \frac{1}{\omega - \gamma - \epsilon_{k-q} + i\delta_{k-q}}. \tag{10.43}$$

Despite the fact that, excepting for the oyster, all diagrams in (10.32) which we added to get this result were infinite, (10.43) is finite! This is due to the fact that unlike the bare interaction which goes to ∞ as $\mathbf{q} \to 0$, the effective interaction remains finite as $\mathbf{q} \to 0$, as shown in (10.41).

Let us first examine (10.43) in the simple limit where we use the static approximation (10.41) for V_{eff}. With the aid of Table 9.1 we find that

$$-i \sum_{\text{RPA}} (\mathbf{k}, \omega) = -i \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left(\frac{4\pi e^2}{q^2 + \lambda^2}\right) \int \frac{d\gamma}{2\pi} iG_0(\mathbf{k} - \mathbf{q}, \omega - \gamma)$$

$$= -i \int_{|\mathbf{k}-\mathbf{q}| < k_F} \frac{d^3\mathbf{q}}{(2\pi)^3} \left(\frac{4\pi e^2}{q^2 + \lambda^2}\right)(-1)$$

$$= -i \int_{|\mathbf{l}| < k_F} \frac{d^3\mathbf{l}}{(2\pi)^3} \frac{4\pi e^2}{[(\mathbf{k} - \mathbf{l})^2 + \lambda^2]}. \tag{10.44}$$