Physics 216: Topics in many-body theory, Spring 2016

Problem set 3: assigned 3/3/16, due 3/17/16

1. Green's function: Define the *retarded* Green's function through

$$iG^r_{\alpha\beta}(t_1, r_1, t_2, r_2) = \begin{cases} \langle \Psi_\alpha(t_1, r_1)\Psi^\dagger_\beta(t_2, r_2) + \Psi^\dagger_\beta(t_2, r_2)\Psi_\alpha(t_1, r_1)\rangle & \text{if } t_1 - t_2 > 0 \\ 0 & \text{if } t_1 - t_2 < 0. \end{cases} \tag{1}$$

Calculate the Fourier-transformed Green's function $G^r(\omega, p)$ for the free Fermi gas at temperature T (i.e., the expectation value $\langle\rangle$ is taken with respect to the finite-temperature Fermi gas). Be sure to specify where any poles occur. Check that contour integration of your result indeed gives zero for negative $t = t_1 - t_2$.

2. Consider two identical 1D non-relativistic bosons of mass m moving on a ring of length L: $x_1, x_2 \in [0, L)$ and

$$H = \frac{\hbar^2}{2m^2}\left(\frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2}\right). \tag{2}$$

(a) What are the energies of the ground state and first excited states? (b) Suppose a repulsive interaction of the form

$$H' = c\delta(r_1 - r_2) \tag{3}$$

is added ($c > 0$). Construct the ground-state bosonic wavefunction by any means you find convenient and find the ground state energy in the limit $c \to \infty$. You do not have to solve the problem for finite c unless you want to.

3. Background for quantum Hall: consider a two-dimensional electron moving in a constant perpendicular magnetic field. Show that in the rotationally symmetric gauge for A, the lowest Landau level eigenstates $\psi_m \sim z^m e^{-|z|^2/4\ell^2}$ with $z = (x + iy)$ and m a nonnegative integer are indeed eigenstates for a particular choice of ℓ (independent of m), and calculate the normalization.

Now suppose that the system of N noninteracting LLL electrons is put in a weak radial confining potential $V(r) = \alpha r^2$, with α sufficiently small that no mixing occurs between Landau levels. The ground state of the N electrons is now a Slater determinant of the single-particle eigenstates $m = 0, \ldots, N - 1$.

Now consider the low-energy edge excitations of this state. For example, moving the last electron out by angular momentum $\hbar$ gives an excited state. How many excitations within the LLL are there of total momentum M, for $M = 1, \ldots, 5$? This is sometimes referred to as the number of "partitions" of the integer M. Give an estimate of the velocity of excitations at the edge.

4. Read Auerbach's description of spin coherent states and do problems 7.4.1 and 7.4.2.

Choose your own adventure: For the last problem, I made one (5A) that is for people who would like some practice with momentum-space diagrammatic perturbation theory, and one set (5B) for people who would rather do a simpler approach to the same physics. My lecture notes are very far from a textbook on diagrammatic many-body physics, so for 5A I recommend you take a look at Mahan, "Many-Particle Physics", or the book by Abrikosov, Gorkov, and Dzyaloshinskii, or any other classic text.

5A. As a warm-up, check the diagrammatic technique discussed in class, write integrals for the first two terms ("Hartree" and "exchange") for $G(\omega, k)$ in momentum-space perturbation theory in the interaction strength V. The diagrammatic representation of these terms is shown on the left side of the scanned page with Feynman rules attached to the node "Diagrammatics": after the bare line, the first is the Hartree ("bubble") and the second is exchange.

Use bubble diagrams ("RPA") as in the attachment to calculate simple Thomas-Fermi screening of the electron-electron interaction (that is, $\omega = 0$ and small k/k_F). For reference, Abrikosov et al. (AGD) and Mahan both have a very complete calculation of this ("Lindhard") screening, but all you need to do is the Thomas-Fermi limit. You may wish to start by reviewing the undergraduate calculation of TF screening, as found in Ashcroft and Mermin or Ziman: the screened interaction is

$$\tilde{V}(\mathbf{q}) = \frac{4\pi e}{q^2 + {k_0}^2}, \quad {k_0}^2 = 4\pi e^2 \frac{\partial n_0}{\partial \mu}. \tag{4}$$

This problem does require a bit of algebra (why it wasn't done in class), but doing one such calculation may be good for you.

5B. Consider the screening of the Coulomb interaction in a metal. The Thomas-Fermi approximation in quantum mechanics, which you can probably find in your QM textbook, consists of computing the electronic density at each point in space in a potential $V(r)$ by using the macroscopic density for a constant chemical potential. Use this idea to write a Poisson equation of the form

$$\nabla^2 V = 4\pi e \delta(\mathbf{r}) + (\text{density response}) \tag{5}$$

and reproduce the formula in (4) for screening of a Coulomb potential.