Physics 216: Topics in many-body theory, spring 2016

Problem set 5: assigned 4/19/16, due 4/28/16

1. Consider the AKLT wavefunction for a spin-one ring that was discussed in class and in Auerbach as a model "valence bond solid":

$$|\Psi\rangle = \prod_{\langle ij\rangle} (a_i^\dagger b_j^\dagger - b_i^\dagger a_j^\dagger)|\mathrm{vac}\rangle. \tag{1}$$

Here a and b are Schwinger bosons, $|\mathrm{vac}\rangle$ is the ground state of the bosons, and the sum is over bonds, with each bond counted only once so that there are two Schwinger bosons per site, as there should be.

It turns out that this wavefunction is a nice example of concepts like matrix product state and quantum entanglement. Consider a ring of three sites for simplicity. The Wikipedia article "AKLT model", applied to this case, states that the AKLT state is a simple matrix product state of the form

$$|\Psi\rangle = \sum_{s_1,s_2,s_3=-1}^{1} \sum_{\alpha\beta\gamma=1}^{2} A_{s_1}^{\alpha\beta} A_{s_2}^{\beta\gamma} A_{s_3}^{\gamma\alpha} |s_1 s_2 s_3\rangle, \tag{2}$$

and an example of the A matrices is given in that article. (Such matrix product states are the type used by the DMRG algorithm to model general states, so it is nice to have a nontrivial example where they are exact.) See whether this form indeed holds for 3 spins.

2. A quick quantum impurity question: starting from the expression in the class notes

$$\begin{aligned} Tr\, G^+(\epsilon) &= \sum_{\mathbf{k}} G^+_{\mathbf{k},\mathbf{k}}(\epsilon) + G^+_{d,d}(\epsilon) \\ &= \sum_{\mathbf{k}} \frac{1}{\epsilon + i\eta - \epsilon_{\mathbf{k}}} + \frac{\partial}{\partial\epsilon} \log\left(\epsilon + i\eta - \epsilon_d - \sum_{\mathbf{k}} \frac{|V_{\mathbf{k}}|^2}{\epsilon + i\eta - \epsilon_{\mathbf{k}}}\right), \end{aligned} \tag{3}$$

use this expression to calculate for yourself the phase shift $\eta(\epsilon)$ and hence the change in the density of states induced by the impurity, which should be Lorentzian (you may assume that ϵ_d and $\tilde{\epsilon}_d$ are near the band center, and hence not within the level width Δ of a band boundary).

3. A one-site example of the quantum-classical mapping: show that a single $O(2)$ quantum rotor, whose Hilbert space is functions on the circle parametrized by θ, with Hamiltonian

$$H_Q = -\Delta \frac{\partial^2}{\partial\theta^2} - \tilde{h}\cos\theta \tag{4}$$

describes the scaling limit of the classical 1D XY model. Do this by writing the XY model partition function as

$$\begin{aligned} Z &= \int_0^{2\pi} \prod_{i=1}^{M} \frac{d\theta_i}{2\pi} \langle\theta_1|T|\theta_2\rangle\langle\theta_2|T|\theta_3\rangle\langle\theta_3|T|\theta_4\rangle \ldots \langle\theta_M|T|\theta_1\rangle \\ &= \mathrm{Tr}\, T^M \end{aligned} \tag{5}$$

with

$$\langle\theta|T|\theta'\rangle = \exp\left(K\cos(\theta-\theta') + \frac{h}{2}(\cos\theta + \cos\theta')\right), \tag{6}$$

and show $\exp(-aH_Q) \approx T$. Find Δ and $\tilde{h}$ in terms of the coupling K, the lattice spacing a, and the classical field h, in the scaling limit where the classical correlation length is much longer than the lattice spacing. The early part of the book of Sachdev on quantum phase transitions may be helpful if you get stuck.